THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics 2020-2021 Term 1
Homework Assignment 3
Due Date: 23 November 2020 (Monday)

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It is also understood that assignments without a properly signed declaration by the student concerned will not be graded by the course teacher.

Signature ______________________ Date ______________________

General Regulations

- All assignments will be submitted and graded on Gradescope. You can view your grades and submit regrade requests here as well. For submitting your PDF homework on Gradescope, here are a few tips.
- Late assignments will receive a grade of 0.
- Print out the cover sheet (i.e. the first page of this document), and sign and date the statement of Academic Honesty.
- Write your COMPLETE name and student ID number legibly on the cover sheet (otherwise we will not take any responsibility for your assignments). Please write your answers using a black or blue pen, NOT any other color or a pencil.
- Write your solutions on A4 white paper. Please do not use any colored paper and make sure that your written solutions are a suitable size (easily read). Failure to comply with these instructions will result in a 10-point deduction).
- Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your answers getting good marks on this homework. Neatness and organization are also essential.

1. Let $f(x) = |x|^3$ on $[-2, 1]$.

   (a) Is Lagrange's mean value theorem applicable to f on the interval $[-2, 1]$?

   (b) If your answer to part (a) is yes, find all possible values $c \in (-2, 1)$, such that

$$\frac{f(1) - f(-2)}{1 - (-2)} = f'(c).$$

Solution

(a) Note that

$$f(x) = \begin{cases} x^3 & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \\ -x^3 & \text{if } -2 \leq x < 0. \end{cases}$$

Note that f is polynomial on $(-2, 0)$ and $(0, 1)$, hence f is differentiable on $(-2, 0) \cup (0, 1)$ with

$$f'(x) = \begin{cases} 3x^2 & \text{if } 0 < x < 1 \\ -3x^2 & \text{if } -2 < x < 0. \end{cases}$$

Note that

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h^3 - 0}{h} = \lim_{h \to 0^+} h^2 = 0$$

and

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h^3 - 0}{h} = \lim_{h \to 0^-} -h^2 = 0,$$

hence,

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

exists and equals to 0, which implies f is differentiable at 0 with $f'(0) = 0$. Then, since f is differentiable on $(-2, 1)$, we must have f is continuous on $(-2, 1)$. Note that

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} x^3 = 1 = f(1)$$

and

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} -x^3 = 8 = f(2).$$

Hence, f is continuous on $[-2, 1]$. Therefore, we can apply Lagrange's mean value theorem to f on $[-2, 1]$.

(b) Note that

$$\frac{f(1) - f(-2)}{1 - (-2)} = \frac{1 - 8}{3} = \frac{-7}{3} < 0,$$

by the definition of f', since x^2 must be non-negative, so the choice of c need to be in $(-2, 0)$, hence $-3c^2 = \frac{-7}{3}$, that is $c = -\frac{\sqrt{7}}{3}$.

2. By using Lagrange's mean value theorem, or otherwise, show that

(a) $\dfrac{x}{1+x} < \ln(1+x) < x$ for $x > 0$;

(b) $ny^{n-1}(x-y) < x^n - y^n < nx^{n-1}(x-y)$ for $n > 1$ and $0 < y < x$.

Solution

(a) Fixed any $x > 0$ (that means the value of x will NOT be change), define $f : [1, 1+x] \to \mathbb{R}$ by

$$f(w) = \ln w$$

for any $w \in [1, 1+x]$. Note that f is a logarithm function on $[1, 1+x]$ and hence continuous on $[1, 1+x]$ and differentiable on $(1, 1+x)$ with

$$f'(w) = \frac{1}{w}$$

for any $w \in (1, 1+x)$. Using Lagrange's mean value theorem, there exist some $\xi \in (1, 1+x)$, such that

$$f'(\xi) = \frac{f(1+x) - f(x)}{1+x-1},$$

that is,

$$\frac{1}{\xi} = \frac{\ln(1+x)}{x}.$$

Since $1 < \xi < 1+x$, we have $\dfrac{1}{1+x} < \dfrac{1}{\xi} < 1$, and hence

$$\frac{1}{1+x} < \frac{\ln(1+x)}{x} < 1,$$

and since $x > 0$, we finally have

$$\frac{x}{1+x} < \ln(1+x) < x.$$

(b) Fixed any $n > 1$ and $x, y > 0$ with $x > y$, define $f : [y, x] \to \mathbb{R}$ by

$$f(w) = w^n$$

for any $w \in [y, x]$. Note that f is a polynomial on $[y, x]$ and hence continuous on $[y, x]$ and differentiable on (x, y) with

$$f'(w) = nw^{n-1}$$

for any $w \in (y, x)$. Using Lagrange's mean value theorem, there exist some $\xi \in (y, x)$, such that

$$f'(\xi) = \frac{f(x) - f(y)}{x - y},$$

that is,

$$n\xi^{n-1} = \frac{x^n - y^n}{x - y}.$$

Since $y < \xi < x$ and $n - 1 > 0$, we have $ny^{n-1} < n\xi^{n-1} < nx^{n-1}$, and hence

$$ny^{n-1} < \frac{x^n - y^n}{x - y} < nx^{n-1},$$

and since $x - y > 0$, we finally have

$$ny^{n-1}(x - y) < x^n - y^n < nx^{n-1}(x - y).$$

3. Let $0 < a < b < \dfrac{\pi}{2}$. Prove that there exists $a < \xi < b$ such that

$$\ln\left(\frac{\cos a}{\cos b}\right) = (b-a)\tan\xi.$$

Solution

Fixed any $0 < a < b < \dfrac{\pi}{2}$, define $f : [a,b] \to \mathbb{R}$ by

$$f(x) = \ln\cos x$$

for any $x \in [a,b]$. Note that $\cos x$ is continuous on $[a,b]$, differentiable on (a,b) and

$$\cos b < \cos x < \cos a$$

for $0 < a < x < b < \dfrac{\pi}{2}$. Moreover, $\ln x$ is continuous on $[\cos b, \cos a]$ and differentiable on $(\cos b, \cos a)$.

Hence, f is continuous on $[a,b]$ and differentiable on (a,b) with

$$f'(x) = \frac{(\cos x)'}{\cos x} = -\tan x$$

for any $x \in (a,b)$.

Using Lagrange's mean value theorem, there exist some $\xi \in (a,b)$, such that

$$f'(\xi) = \frac{f(a)-f(b)}{a-b},$$

that is,

$$-\tan\xi = \frac{\ln\cos a - \ln\cos b}{a-b}.$$

Since $\ln x - \ln y = \ln(\frac{x}{y})$, we have $\ln\cos a - \ln\cos b = \ln\left(\dfrac{\cos a}{\cos b}\right)$, and hence

$$\ln\left(\frac{\cos a}{\cos b}\right) = (b-a)\tan\xi.$$

4. Let $f(x) = \dfrac{\sin x}{x}$ for $x > 0$. Compute $f'(x)$. Hence, or otherwise, show that $x\sin y > y\sin x$ whenever $0 < y < x \le \pi$.

Solution

By quotient rule, we have

$$f'(w) = \frac{w\cos w - \sin w}{w^2}$$

for any $w > 0$. Since w^2 is non-negative, to study the sign of f', it suffices to study the sign of

$$g(w) = w\cos w - \sin w.$$

Note that

$$g'(w) = -w \sin w + \cos w - \cos w = -w \sin w$$

and hence $g'(w) < 0$ for any $w \in (0, \pi)$, that is g is strictly decreasing on $(0, \pi)$. Note that g is still continuous at $w = 0$ and $w = \pi$, so we have

$$g(w) < g(0) = 0$$

for any $w \in (0, \pi]$. Therefore,

$$f'(w) = \frac{g(w)}{w^2} < 0$$

for any $w \in (0, \pi]$. Thus, f is strictly decreasing on $(0, \pi]$. If $0 < y < x \leq \pi$, then

$$f(x) < f(y)$$

which implies

$$\frac{\sin x}{x} < \frac{\sin y}{y}.$$

Since $x, y > 0$, we finally have

$$y \sin x < x \sin y.$$

5. Evaluate the following limits.

(a) $\lim_{x\to 0} \dfrac{\sin^{-1} x - \tan^{-1} x}{x^3}$

(b) $\lim_{x\to 0} \log_{\tan x}(\tan 2x)$

(c) $\lim_{x\to 0^+} \tan x \ln \sin x$

(d) $\lim_{x\to 1} \left(\dfrac{1}{\ln x} - \dfrac{x}{x-1}\right)$

(e) $\lim_{x\to +\infty} \dfrac{e^{1+\ln x}}{\ln(1+e^x)}$

Solution

(a) We compute the Taylor series of $\sin^{-1} x$ and $\tan^{-1} x$ at $x = 0$ to the third order:

$$(\sin^{-1})'(x) = (1-x^2)^{\frac{-1}{2}}$$
$$(\sin^{-1})''(x) = x(1-x^2)^{\frac{-3}{2}}$$
$$(\sin^{-1})'''(x) = (1+2x^2)(1-x^2)^{\frac{-5}{2}}$$

$$(\tan^{-1})'(x) = (1+x^2)^{-1}$$
$$(\tan^{-1})''(x) = -2x(1+x^2)^{-2}$$
$$(\tan^{-1})'''(x) = 2(3x^2-1)(1+x^2)^{-3}$$

So the Taylor series are

$$\sin^{-1}(x) = x + \frac{x^3}{6} + O(x^4)$$

and

$$\tan^{-1}(x) = x - \frac{x^3}{3} + O(x^4)$$

Hence the limit is

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin^{-1} x - \tan^{-1} x}{x^3} &= \lim_{x\to 0} \frac{(x + \frac{1}{6}x^3 + O(x^4)) - (x - \frac{1}{3}x^3 + O(x^4))}{x^3} \\
&= \lim_{x\to 0} \frac{\frac{1}{2}x^3 + O(x^4)}{x^3} \\
&= \frac{1}{2}
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x\to 0} \log_{\tan x}(\tan 2x) &= \lim_{x\to 0} \frac{\ln \tan 2x}{\ln \tan x} \\
&= \lim_{x\to 0} \frac{(\ln \tan 2x)'}{(\ln \tan x)'} \\
&= \lim_{x\to 0} 2\frac{\tan x \cos^2 x}{\tan 2x \cos^2 2x} \\
&= \lim_{x\to 0} 2\frac{\sin 2x}{\sin 4x} \\
&= \lim_{x\to 0} \frac{\sin 2x}{2x} \lim_{x\to 0} \frac{4x}{\sin 4x} \\
&= 1
\end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to 0^+} \tan x \ln \sin x &= \lim_{x\to 0^+} \frac{\ln \sin x}{\cot x}\\ &= \lim_{x\to 0^+} \frac{(\ln \sin x)'}{(\cot x)'}\\ &= \lim_{x\to 0^+} \frac{\frac{1}{\sin x}\cos x}{-\csc^2 x}\\ &= \lim_{x\to 0^+} -\sin x \cos x = 0\end{aligned}$$

(d)

$$\begin{aligned}\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{x}{x-1}\right) &= \lim_{x\to 1} \frac{x-1-\ln x}{(x-1)\ln x} - 1\\ &= \lim_{x\to 1} \frac{(x-1-\ln x)'}{((x-1)\ln x)'} - 1\\ &= \lim_{x\to 1} \frac{1-\frac{1}{x}}{\ln x + \frac{x-1}{x}} - 1\\ &= \lim_{x\to 1} \frac{x-1}{x\ln x + x - 1} - 1\\ &= \lim_{x\to 1} \frac{(x-1)'}{(x\ln x + x - 1)'} - 1\\ &= \lim_{x\to 1} \frac{1}{\ln x + 1 + 1} - 1\\ &= -\frac{1}{2}\end{aligned}$$

(e)

$$\begin{aligned}\lim_{x\to+\infty} \frac{e^{1+\ln x}}{\ln(1+e^x)} &= \lim_{x\to+\infty} \frac{xe}{\ln(1+e^x)}\\ &= \lim_{x\to+\infty} \frac{(xe)'}{(\ln(1+e^x))'}\\ &= \lim_{x\to+\infty} \frac{e}{\frac{1}{1+e^x}e^x}\\ &= \lim_{x\to+\infty} e(1+e^{-x}) = e\end{aligned}$$

6. Evaluate the following limits.

(a) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

(b) $\lim_{x\to 1} x^{\frac{2x}{x-1}}$

(c) $\lim_{x\to 0}\frac{(1+x)^x-1}{x^2}$

(d) $\lim_{x\to +\infty}\left(\frac{x^2-2x+1}{x^2-4x+2}\right)^x$

Solution

(a)

$$\begin{aligned}
\lim_{x\to 0}\frac{1}{x^2}\ln\frac{\sin x}{x} &= \lim_{x\to 0}\frac{(\ln\frac{\sin x}{x})'}{(x^2)'}\\
&= \lim_{x\to 0}\frac{\frac{x}{\sin x}(\frac{\cos x}{x}-\frac{\sin x}{x^2})}{2x}\\
&= \frac{1}{2}\lim_{x\to 0}\frac{x\cos x-\sin x}{x^2\sin x}\\
&= \frac{1}{2}\lim_{x\to 0}\frac{(x\cos x-\sin x)'}{(x^2\sin x)'}\\
&= \frac{1}{2}\lim_{x\to 0}\frac{-x\sin x}{2x\sin x+x^2\cos x}\\
&= \frac{1}{2}\lim_{x\to 0}\frac{-1}{2+\frac{x}{\tan x}}\\
&= \frac{1}{2}\frac{-1}{2+1}=-\frac{1}{6}
\end{aligned}$$

So

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} &= e^{\lim_{x\to 0}\frac{1}{x^2}\ln\frac{\sin x}{x}}\\
&= e^{\frac{-1}{6}}
\end{aligned}$$

(b)

$$\begin{aligned}
\lim_{x\to 1}\frac{2x}{x-1}\ln x &= 2\lim_{x\to 1}\frac{\ln x}{1-\frac{1}{x}}\\
&= 2\lim_{x\to 1}\frac{(\ln x)'}{(1-\frac{1}{x})'}\\
&= 2\lim_{x\to 1}\frac{\frac{1}{x}}{\frac{1}{x^2}}=2
\end{aligned}$$

So

$$\begin{aligned}
\lim_{x\to 1}x^{\frac{2x}{x-1}} &= e^{\lim_{x\to 1}\frac{2x}{x-1}\ln x}\\
&= e^2
\end{aligned}$$

(c) We compute the Taylor series of $f(x) = (1+x)^x = e^{x\ln(1+x)}$ at $x = 0$ up to x^2:

$$f'(x) = e^{x\ln(1+x)}(\ln(1+x) + 1 - \frac{1}{1+x})$$

$$f''(x) = e^{x\ln(1+x)}(\ln(1+x) + 1 - \frac{1}{1+x})^2 + e^{x\ln(1+x)}\frac{x+2}{(1+x)^2}$$

As $f(0) = e^{0\ln 1} = 1$, $f'(0) = e^{0\ln 1}(\ln 1 + 1 - \frac{0}{1+0}) = 0$, $f''(0) = e^{0\ln 1}(\ln 1 + 1 - \frac{0}{1+0})^2 + e^{0\ln 1}\frac{0+2}{(1+0)^2} = 2$, we have $(1+x)^x = 1 + x^2 + O(x^3)$, so

$$\lim_{x\to 0}\frac{(1+x)^x - 1}{x^2} = \lim_{x\to 0}\frac{x^2 + O(x^3)}{x^2} = 1$$

(d)

$$\begin{aligned}
\lim_{x\to+\infty} x\ln\frac{(x-1)^2}{x^2-4x+2} &= \lim_{x\to+\infty}\frac{\ln\frac{(x-1)^2}{x^2-4x+2}}{x^{-1}} \\
&= \lim_{x\to+\infty}\frac{(\ln\frac{(x-1)^2}{x^2-4x+2})'}{(x^{-1})'} \\
&= \lim_{x\to+\infty}\frac{2}{-x^{-2}}\frac{-x}{(x-1)(x^2-4x+2)} \\
&= \lim_{x\to+\infty}\frac{2x^3}{(x-1)(x^2-4x+2)} = 2
\end{aligned}$$

So

$$\lim_{x\to+\infty}\left(\frac{x^2-2x+1}{x^2-4x+2}\right)^x = e^{\lim_{x\to+\infty} x\ln\frac{(x-1)^2}{x^2-4x+2}} = e^2$$

7. Find the x-intercepts, y-intercepts, asymptotes if there is any and sketch the graphs of the following functions.

(a) $y = \dfrac{x+5}{x-2}$

(b) $y = \dfrac{x^2-2}{x-1}$

(c) $y = |4+3x-x^2|$

(d) $y = x|x+2|$

(e) $y = \left|\dfrac{7-2x}{x+3}\right|$

(f) $y = \dfrac{1}{|x^2-4|}$

Solution

(a) The x-intercept is at where $y = \frac{x+5}{x-2} = 0$, so the x-intercept is $(-5, 0)$.
The y-intercept is at where $x = 0$, so the y-intercept is $(0, \frac{0+5}{0-2}) = (0, -\frac{5}{2})$.
At $x = 2$, the denominator becomes 0, so $x = 2$ is a vertical asymptote.
Since $\lim_{x\to\pm\infty} \frac{y(x)}{x} = 0$ and $\lim_{x\to\pm\infty} y(x) = 1$, $y = 1$ is a horizontal asymptote.

(b) The x-intercept is at where $y = \frac{x^2-2}{x-1} = 0$, so the x-intercepts are $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$.
The y-intercept is at where $x = 0$, so the y-intercept is $(0, \frac{0^2-2}{0-1}) = (0, 2)$.
At $x = 1$, the denominator becomes 0, so $x = 1$ is a vertical asymptote.
Since $\lim_{x\to\pm\infty} \frac{y(x)}{x} = 1$ and $\lim_{x\to\pm\infty} y(x) - x = 1$, $y = x + 1$ is an oblique asymptote.

(c) The x-intercept is at where $y = |4+3x-x^2| = 0$, so the x-intercepts are $(-1, 0)$ and $(4, 0)$.
The y-intercept is at where $x = 0$, so the y-intercept is $(0, |4+3\cdot 0-0^2|) = (0, 4)$.
Since the function has no singularity and $\lim_{x\to\pm\infty} \frac{y(x)}{x} = \pm\infty$, the function has no asymptote.

(d) The x-intercept is at where $y = x|x+2| = 0$, so the x-intercepts are $(0, 0)$ and $(-2, 0)$.
The y-intercept is at where $x = 0$, so the y-intercept is $(0, 0 \cdot |0+2|) = (0, 0)$.
Since the function has no singularity and $\lim_{x\to\pm\infty} \frac{y(x)}{x} = +\infty$, the function has no asymptote.

(e) The x-intercept is at where $y = \left|\frac{7-2x}{x+3}\right| = 0$, so the x-intercept is $(\frac{7}{2}, 0)$.
The y-intercept is at where $x = 0$, so the y-intercept is $(0, \left|\frac{7-2\cdot 0}{0+3}\right|) = (0, \frac{7}{3})$.
At $x = -3$, the denominator becomes 0, so $x = -3$ is a vertical asymptote.
Since $\lim_{x\to\pm\infty} \frac{y(x)}{x} = 0$ and $\lim_{x\to\pm\infty} y(x) = 2$, $y = 2$ is a horizontal asymptote.

(f) The x-intercept is at where $y = \frac{1}{|x^2-4|} = 0$, so the function has no x-intercept.
The y-intercept is at where $x = 0$, so the y-intercept is $(0, \frac{1}{|0^2-4|}) = (0, \frac{1}{4})$.
At $x = -2$ and at $x = 2$, the denominator becomes 0, so $x = 2$ and $x = -2$ are vertical asymptotes.
Since $\lim_{x\to\pm\infty} \frac{y(x)}{x} = 0$ and $\lim_{x\to\pm\infty} y(x) = 0$, $y = 0$ is a horizontal asymptote.

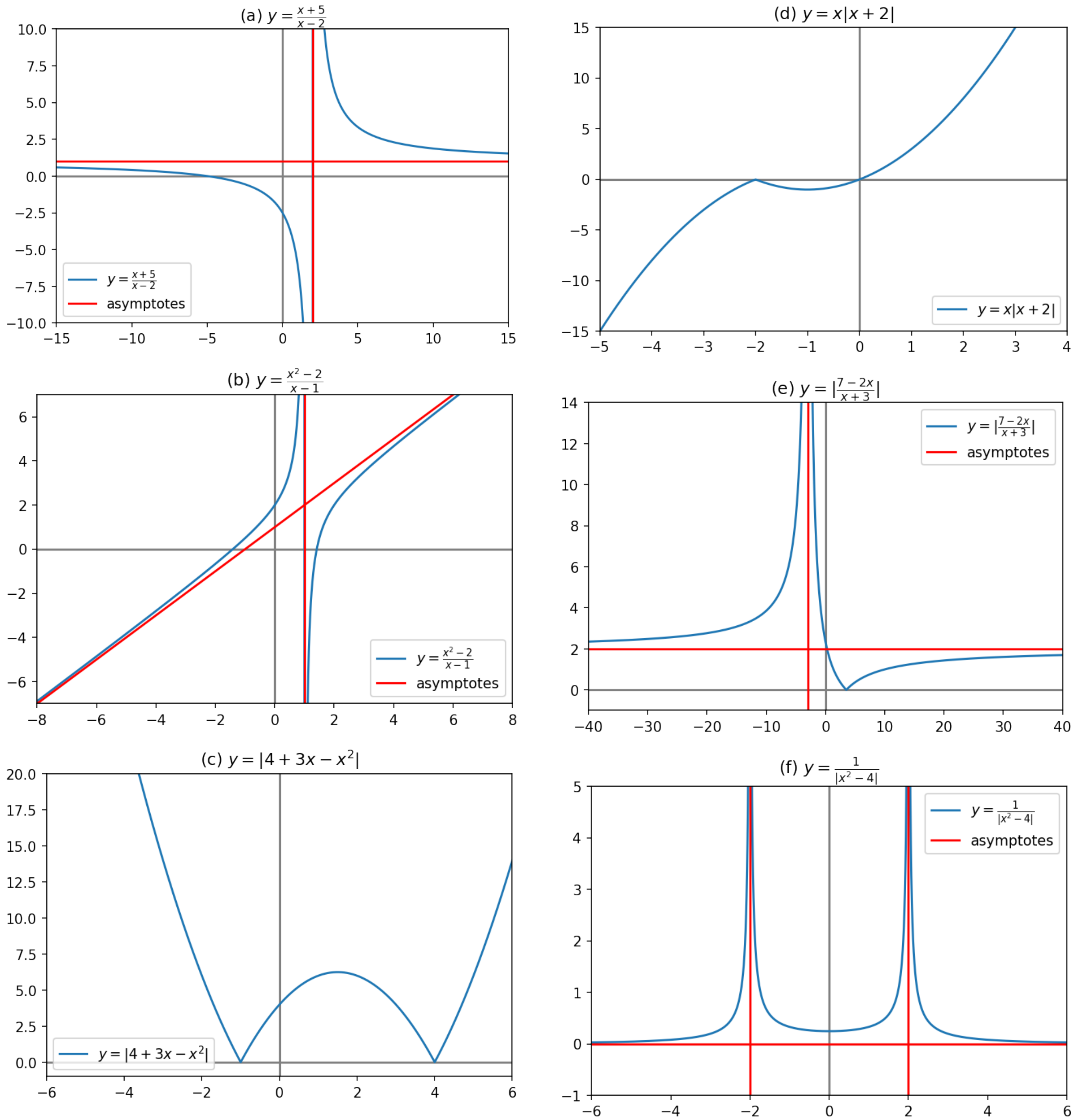

Figure 1: The graphs of the functions for question 7. Asymptotes, if they exist, are also drawn.

8. For each of the following functions $f(x)$, find

- $f'(x)$ and $f''(x)$.
- range of values of x for which $f(x)$ is increasing.
- asymptotes of $y = f(x)$.
- all relative extremum points

Then sketch the graph of $y = f(x)$.

(a) $f(x) = \frac{x}{(x-2)^2}$

(b) $f(x) = \frac{x^2+5x+7}{x+2}$

(c) $f(x) = \frac{x^2}{x^2-2x+2}$

(d) $f(x) = x^{\frac{2}{3}} - 1$

Solution

(a)

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{x}{(x-2)^2} = \frac{1}{(x-2)^2} - \frac{2x}{(x-2)^3} = -\frac{x+2}{(x-2)^3}$$

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x} - \frac{x+2}{(x-2)^3} = -\left(\frac{1}{(x-2)^3} - \frac{3(x+2)}{(x-2)^4}\right) = \frac{2x+8}{(x-2)^4}$$

f is differentiable on the domain $(-\infty, 2) \cup (2, \infty)$, and $f'(x) > 0$ if and only if $-2 < x < 2$. So f is increasing on $[-2, 2)$.

Since when $x = 2$, the denominator becomes 0, so $x = 2$ is a vertical asymptote.

As $\lim_{x\to\pm\infty} \frac{f(x)}{x} = 0$ and $\lim_{x\to\pm\infty} f(x) = 0$, $y = 0$ is a horizontal asymptote.

The only critical point of $f(x)$ is $x = -2$, at which $f''(-2) = \frac{1}{64} > 0$, so $x = -2$ is the only relative extremum and is a relative minimum.

(b)

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2+5x+7}{x+2} = \frac{2x+5}{x+2} - \frac{x^2+5x+7}{(x+2)^2} = \frac{x^2+4x+3}{(x+2)^2} = \frac{(x+1)(x+3)}{(x+2)^2}$$

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2+4x+3}{(x+2)^2} = \frac{2x+4}{(x+2)^2} - (x^2+4x+3)\frac{-2}{(x+2)^3} = \frac{2}{(x+2)^3}$$

f is differentiable on the domain $(-\infty, -2) \cup (-2, \infty)$, and $f'(x) > 0$ if and only if $x < -3$ or $-1 < x$. Also, $f(-3) = -1 < 3 = f(-1)$. So f is increasing on $(-\infty, -3] \cup [-1, \infty)$

Since when $x = -2$, the denominator becomes 0, so $x = -2$ is a vertical asymptote.

As $\lim_{x\to\pm\infty} \frac{f(x)}{x} = 1$ and $\lim_{x\to\pm\infty} f(x) - x = 3$, so $y = x + 3$ is an oblique asymptote.

The only critical points are $x = -1$ and $x = -3$. Since $f''(-1) = 2 > 0$ and $f''(-3) = -2 < 0$, so the only relative extrema are at $x = -1$ and $x = -3$, where $x = -1$ is a relative minimum and $x = -3$ is a relative maximum.

(c)

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2}{x^2-2x+2} = \frac{2x}{x^2-2x+2} - \frac{x^2(2x-2)}{(x^2-2x+2)^2} = -\frac{2x(x-2)}{(x^2-2x+2)^2}$$

$$f''(x) = \frac{-4x+4}{(x^2-2x+2)^2} - \frac{2(-2x^2+4x)(2x-2)}{(x^2-2x+2)^3} = \frac{4(x-1)(x^2-2x-2)}{(x^2-2x+2)^3}$$

f is differentiable on the domain $(-\infty, \infty)$, and $f'(x) > 0$ if and only if $0 < x < 2$, so f is increasing on $[0, 2]$.

As $\lim_{x\to\pm\infty} \frac{f(x)}{x} = 0$ and $\lim_{x\to\pm\infty} f(x) = 1$, $y = 1$ is a horizontal asymptote.

The critical points of f are $x = 0$ and $x = 2$. Since $f''(0) = 1 > 0$ and $f''(2) = -1 < 0$, so the only relative extrema are $x = 0$ and $x = 2$, where $x = 0$ is a relative minimum and $x = 2$ is a relative maximum.

(d)

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(x^{\frac{2}{3}} - 1) = \frac{2}{3}x^{\frac{-1}{3}} = \frac{2}{3\sqrt[3]{x}}$$

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{2}{3}x^{\frac{-1}{3}} = -\frac{2}{9}x^{\frac{-4}{3}} = -\frac{2}{9\sqrt[3]{x^4}}$$

f is differentiable on $(-\infty, 0) \cup (0, \infty)$, and $f'(x) > 0$ if and only if $x > 0$. So f is increasing on $[0, \infty)$

Since $\lim_{x\to\pm\infty} \frac{f(x)}{x} = 0$ but $\lim_{x\to\pm\infty} f(x)$ does not exist. So f has no asymptote.

The only critical points of f are $x = 0$ as f is not differentiable at $x = 0$ and $f'(x) \neq 0$ on $(-\infty, 0) \cup (0, \infty)$. Since for $x \neq 0$, $f(x) = -1 + \sqrt[3]{x^2} \geq -1 = f(0)$, $x = 0$ is the only relative extremum and is a relative minimum.

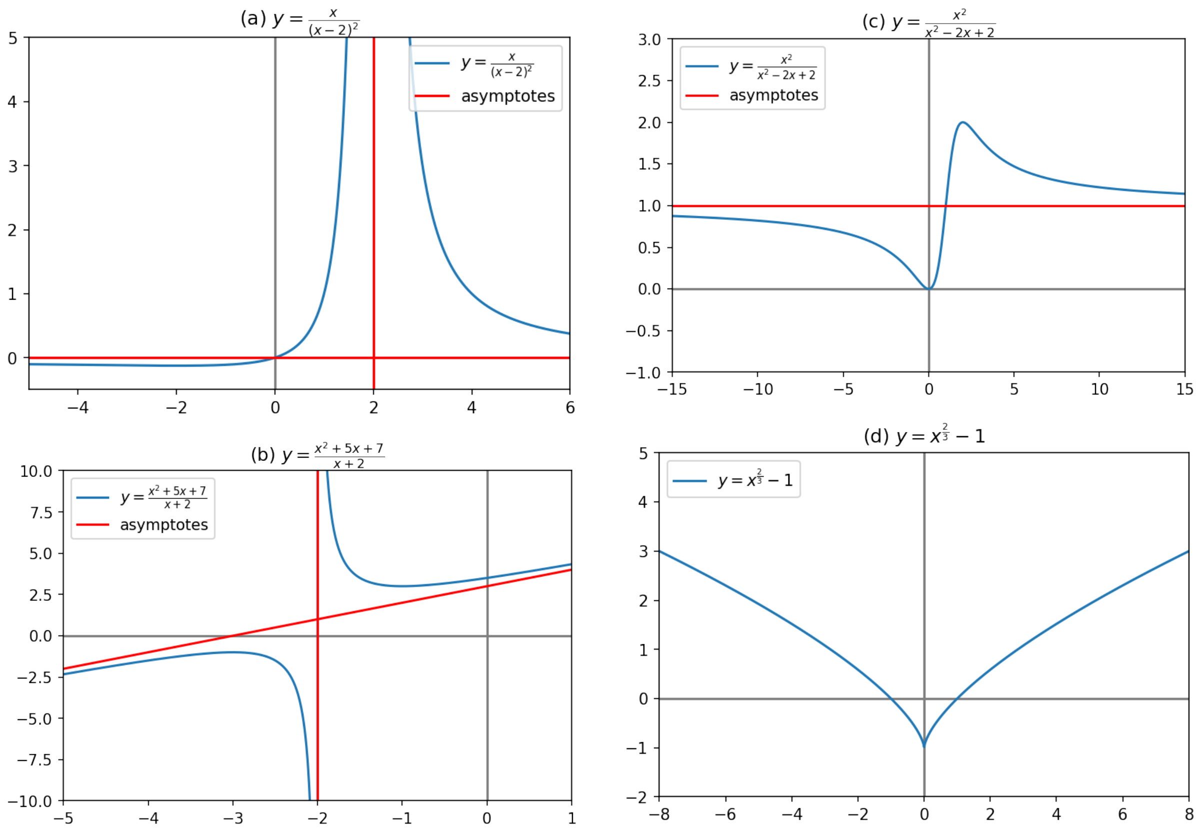

Figure 2: The graphs of the functions for question 8. Asymptotes, if they exist, are also drawn.

9. For each of the following functions $f(x)$, find $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0)$ and the Taylor series up to the term in x^3 of $f(x)$ about the point $x = 0$.

(a) $f(x) = \ln \cos x$ (b) $f(x) = e^x \sin x$

Solution

(a) $f(0) = \ln \cos 0 = 0$

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \ln \cos x = \frac{1}{\cos x}(-\sin x) = -\tan x$$

So $f'(0) = -\tan 0 = 0$

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x} - \tan x = -\sec^2 x$$

So $f''(0) = -\sec^2 0 = -1$

$$f'''(x) = \frac{\mathrm{d}}{\mathrm{d}x} - \sec^2 x = \frac{2}{\cos^3}(-\sin x) = -2 \tan x \sec^2 x$$

So $f'''(0) = 0$

So the Taylor series of $f(x) = \ln \cos x$ about $x = 0$ up to x^3 is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + O(x^4) = -\frac{1}{2}x^2 + O(x^4)$$

(b) $f(0) = e^0 \sin 0 = 0$

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} e^x \sin x = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$$

So $f'(0) = e^0(\sin 0 + \cos 0) = 1$

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x} e^x(\sin x + \cos x) = e^x(\sin x + \cos x) + e^x(\cos x - \sin x) = 2e^x \cos x$$

So $f''(0) = 2e^0 \cos 0 = 2$

$$f'''(0) = \frac{\mathrm{d}}{\mathrm{d}x} 2e^x \cos x = 2(e^x \cos x - e^x \sin x) = 2e^x(\cos x - \sin x)$$

So $f'''(0) = 2e^0(\cos 0 - \sin 0) = 2$

So the Taylor series of $f(x) = e^x \sin x$ about $x = 0$ up to x^3 is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + O(x^4) = x + x^2 + \frac{1}{3}x^3 + O(x^4)$$

10. Find the Taylor series up to the term in $(x-c)^3$ of the functions about $x = c$.

(a) $\dfrac{1}{1+x}$; $c = 1$.

(b) $\dfrac{2-x}{3+x}$; $c = 1$.

(c) $\dfrac{x}{(x-1)(x-2)}$; $c = 0$.

(d) $\cos x$; $c = \frac{\pi}{4}$.

(e) $\sin^2 x$; $c = 0$

(f) $\ln x$; $c = e$.

(g) 3^x; $c = 0$.

(h) $\sqrt{2+x}$; $c = 1$.

(i) $\dfrac{1}{\sqrt{7-3x}}$; $c = 1$.

Solution

(a) Let $f(x) = \frac{1}{1+x}$. Then $f(c) = \frac{1}{1+c} = \frac{1}{2}$, $f'(c) = \frac{-1}{(1+c)^2} = -\frac{1}{4}$, $f''(c) = \frac{2}{(1+c)^3} = \frac{1}{4}$, $f'''(c) = \frac{-6}{(1+c)^4} = -\frac{3}{8}$.
So $\frac{1}{1+x} = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$
$= \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3 + O((x-1)^4)$

(b) Let $f(x) = \frac{2-x}{3+x} = -1 + \frac{5}{3+x}$. Then $f(c) = -1 + \frac{5}{3+c} = \frac{1}{4}$, $f'(c) = \frac{-5}{(3+c)^2} = -\frac{5}{16}$, $f''(c) = \frac{10}{(3+c)^3} = \frac{5}{32}$, $f'''(c) = \frac{-30}{(3+c)^4} = -\frac{15}{128}$.
So $\frac{2-x}{3+x} = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$
$= \frac{1}{4} - \frac{5}{16}(x-1) + \frac{5}{64}(x-1)^2 - \frac{5}{256}(x-1)^3 + O((x-1)^4)$

(c) Let $f(x) = \frac{x}{(x-1)(x-2)}$. Then $f(c) = \frac{0}{(0-1)(0-2)} = 0$, $f'(c) = -\frac{c^2-2}{(c-1)^2(c-2)^2} = \frac{1}{2}$, $f''(c) = \frac{2(c^3-6c+6)}{(c-1)^3(c-2)^3} = \frac{3}{2}$, $f'''(c) = -\frac{6(c^4-12c^2+24c-14)}{(c-1)^4(c-2)^4} = \frac{21}{4}$.
So $\frac{x}{(x-1)(x-2)} = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$
$= \frac{1}{2}x + \frac{3}{4}x^2 + \frac{7}{8}x^3 + O(x^4)$

(d) Let $f(x) = \cos x$. Then $f(c) = \cos c = \frac{\sqrt{2}}{2}$, $f'(c) = -\sin c = -\frac{\sqrt{2}}{2}$, $f''(c) = -\cos c = -\frac{\sqrt{2}}{2}$, $f'''(c) = \sin c = \frac{\sqrt{2}}{2}$.
So $\cos x = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$
$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x - \frac{\pi}{4}) - \frac{\sqrt{2}}{4}(x - \frac{\pi}{4})^2 + \frac{\sqrt{2}}{12}(x - \frac{\pi}{4})^3 + O((x - \frac{\pi}{4})^4)$

(e) Let $f(x) = \sin^2 x$. Then $f(c) = \sin^2 c = 0$, $f'(c) = \sin(2c) = 0$, $f''(c) = 2\cos(2c) = 2$, $f'''(c) = -4\sin(2c) = 0$.
So $\sin^2 x = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$
$= x^2 + O(x^4)$

(f) Let $f(x) = \ln x$. Then $f(c) = \ln c = 1$, $f'(c) = \frac{1}{c} = \frac{1}{e}$, $f''(c) = -\frac{1}{c^2} = -\frac{1}{e^2}$, $f'''(c) = \frac{2}{c^3} = \frac{2}{e^3}$.
So $\ln x = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$
$= 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3 + O((x-e)^4)$

(g) Let $f(x) = 3^x$. Then $f(c) = 3^c = 1$, $f'(c) = 3^c \ln 3 = \ln 3$, $f''(c) = 3^c(\ln 3)^2 = (\ln 3)^2$, $f'''(c) = 3^c(\ln 3)^3 = (\ln 3)^3$.
So $3^x = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$
$= 1 + x\ln 3 + \frac{(\ln 3)^2}{2}x^2 + \frac{(\ln 3)^3}{6}x^3 + O(x^4)$

(h) Let $f(x) = \sqrt{2+x}$. Then $f(c) = \sqrt{2+c} = \sqrt{3}$, $f'(c) = \frac{1}{2}(2+c)^{\frac{-1}{2}} = \frac{\sqrt{3}}{6}$, $f''(c) = -\frac{1}{4}(2+c)^{\frac{-3}{2}} = -\frac{\sqrt{3}}{36}$, $f'''(c) = \frac{3}{8}(2+c)^{\frac{-5}{2}} = \frac{\sqrt{3}}{72}$.

So $\sqrt{2+x} = f(x) = f(c)+f'(c)(x-c)+\frac{f''(c)}{2}(x-c)^2+\frac{f'''(c)}{6}(x-c)^3+O((x-c)^4)$
$= \sqrt{3} + \frac{\sqrt{3}}{6}(x-1) - \frac{\sqrt{3}}{72}(x-1)^2 + \frac{\sqrt{3}}{432}(x-1)^3 + O((x-1)^4)$

(i) Let $f(x) = \frac{1}{\sqrt{7-3x}}$. Then $f(c) = \frac{1}{\sqrt{7-3c}} = \frac{1}{2}$, $f'(c) = -\frac{1}{2}(7-3c)^{\frac{-3}{2}} = \frac{3}{16}$, $f''(c) = \frac{27}{3}(7-3x)^{\frac{-5}{2}} = \frac{27}{128}$, $f'''(c) = \frac{405}{8}(7-3x)^{\frac{-7}{2}} = \frac{405}{1024}$.
So $\frac{1}{\sqrt{7-3c}} = f(x) = f(c)+f'(c)(x-c)+\frac{f''(c)}{2}(x-c)^2+\frac{f'''(c)}{6}(x-c)^3+O((x-c)^4)$
$= \frac{1}{2} + \frac{3}{16}(x-1) + \frac{27}{256}(x-1)^2 + \frac{135}{2048}(x-1)^3 + O((x-1)^4)$

Alternatively, by using the Taylor series of the elementary functions,

(a) $\frac{1}{x+1} = \frac{1}{2}\frac{1}{1+\frac{x-1}{2}} = \frac{1}{2}\left(1 - \frac{x-1}{2} + (\frac{x-1}{2})^2 - (\frac{x-1}{2})^3 + O((x-1)^4)\right)$
$= \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3 + O((x-1)^4)$

(b) $\frac{2-x}{3+x} = -1 + \frac{5}{4}\frac{1}{1+\frac{x-1}{4}} = -1 + \frac{5}{4}\left(1 - \frac{x-1}{4} + (\frac{x-1}{4})^2 - (\frac{x-1}{4})^3 + O((x-1)^4)\right)$
$= \frac{1}{4} - \frac{5}{16}(x-1) + \frac{5}{64}(x-1)^2 - \frac{5}{256}(x-1)^3 + O((x-4)^2)$

(c) $\frac{x}{(x-1)(x-2)} = -\frac{1}{1-\frac{x}{2}} + \frac{1}{1-x}$
$= -\left(1 + \frac{x}{2} + (\frac{x}{2})^2 + (\frac{x}{2})^3 + O(x^4)\right) + (1 + x + x^2 + x^3 + O(x^4))$
$= \frac{1}{2}x + \frac{3}{4}x^2 + \frac{7}{8}x^3 + O(x^4)$

(d) $\cos x = \cos(x - \frac{\pi}{4} + \frac{\pi}{4}) = \frac{\sqrt{2}}{2}\left(\cos(x-\frac{\pi}{4}) - \sin(x-\frac{\pi}{4})\right)$
$= \frac{\sqrt{2}}{2}\left((1 - \frac{(x-\frac{\pi}{4})^2}{2} + O((x-\frac{\pi}{4})^4)) - ((x-\frac{\pi}{4}) - \frac{(x-\frac{\pi}{4})^3}{6} + O((x-\frac{\pi}{4})^4))\right)$
$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x-\frac{\pi}{4}) - \frac{\sqrt{2}}{4}(x-\frac{\pi}{4})^2 + \frac{\sqrt{2}}{12}(x-\frac{\pi}{4})^3 + O((x-\frac{\pi}{4})^4)$

(e) $\sin^2 x = \frac{1}{2}(1-\cos(2x)) = \frac{1}{2}(1 - (1 - \frac{(2x)^2}{2} + O(x^4)))$
$= x^2 + O(x^4)$

(f) $\ln x = 1 + \ln(1 + \frac{x-e}{e}) = 1 + \left(\frac{x-e}{e} - \frac{1}{2}(\frac{x-e}{e})^2 + \frac{1}{3}(\frac{x-e}{e})^3 + O((x-e)^4)\right)$
$= 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3 + O((x-e)^4)$

(g) $3^x = e^{x\ln 3} = 1 + x\ln 3 + \frac{1}{2}(x\ln 3)^2 + \frac{1}{6}(x\ln 3)^3 + O(x^4)$
$= 1 + x\ln 3 + \frac{(\ln)^3}{2}x^2 + \frac{(\ln 3)^3}{6}x^3 + O(x^4)$

(h) $\sqrt{2+x} = \sqrt{3}(1 + \frac{x-1}{3})^{\frac{1}{2}}$
$= \sqrt{3}\left(1 + \frac{1}{2}\frac{x-1}{3} + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}(\frac{x-1}{3})^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{6}(\frac{x-1}{3})^3 + O((x-1)^4)\right)$
$= \sqrt{3} + \frac{\sqrt{3}}{6}(x-1) - \frac{\sqrt{3}}{72}(x-1)^2 + \frac{\sqrt{3}}{432}(x-1)^3 + O((x-1)^4)$

(i) $\frac{1}{\sqrt{7-3x}} = \frac{1}{2}(1 - \frac{x-1}{4/3})^{\frac{-1}{2}}$
$= \frac{1}{2}(1 - \frac{-1}{2}\frac{x-1}{4/3} + \frac{\frac{-1}{2}(\frac{-1}{2}-1)}{2}(\frac{x-1}{4/3})^2 - \frac{\frac{-1}{2}(\frac{-1}{2}-1)(\frac{-1}{2}-2)}{6})(\frac{x-1}{4/3})^3 + O((x-1)^4)$
$= \frac{1}{2} + \frac{3}{16}(x-1) + \frac{27}{256}(x-1)^2 + \frac{135}{2048}(x-1)^3 + O((x-1)^4)$

11. Suppose $y = f(x)$ is a function which satisfies $y + \frac{y^3}{3} = x$.

(a) Show that $\frac{d^2y}{dx^2} = -\frac{2y}{(1+y^2)^3}$.

(b) Find the Taylor series up to the term in x^3 of $f(x)$ about the point $x = 0$.

Solution

(a) Differentiate the equation implicitly:

$$\frac{dy}{dx} + y^2\frac{dy}{dx} = 1$$

Then

$$\frac{dy}{dx} = \frac{1}{1+y^2}$$

Differentiate one more time yields:

$$\frac{d^2y}{dx^2} + 2y(\frac{dy}{dx})^2 + y^2\frac{d^2y}{dx^2} = 0$$

Now

$$\begin{aligned}\frac{d^2y}{dx^2} &= -\frac{2y(\frac{dy}{dx})^2}{1+y^2} \\ &= -\frac{2y}{(1+y^2)^3}\end{aligned}$$

(b) Note that $-\frac{d^2y}{dx^2} = 2y(\frac{dy}{dx})^3$,

$$\begin{aligned}-\frac{d^3y}{dx^3} &= 2(\frac{dy}{dx})(\frac{dy}{dx})^3 + 2y\cdot 3(\frac{dy}{dx})^2\frac{d^2y}{dx^2} \\ &= 2(\frac{dy}{dx})^4 + 6y(\frac{dy}{dx})^2\frac{d^2y}{dx^2} \\ &= \frac{2}{(1+y^2)^4} - \frac{6y}{(1+y^2)^2}\frac{2y}{(1+y^2)^3} \\ &= \frac{2-10y^2}{(1+y^2)^5}\end{aligned}$$

Hence

$$f(0) + \frac{f^3(0)}{3} = f(0)(1 + \frac{f^2(0)}{3}) = 0 \implies f(0) = 0$$
$$f'(0) = \frac{1}{1+0^2} = 1, f''(0) = -\frac{2\cdot 0}{(1+0^2)^3} = 0, f'''(0) = -\frac{2-0}{(1+0)^5} = -2.$$

Therefore, the Taylor series of $f(x)$ about the point $x = 0$ is given by

$$x - \frac{1}{3}x^3 + \cdots$$

12. By considering appropriate Taylor series expansions, evaluate the limits below:

(a) $\lim_{x\to 0} \dfrac{e^{2x}-1}{\ln(1+x)}$

(b) $\lim_{x\to 0} \left(\dfrac{1}{\ln(1+x)} + \dfrac{1}{\ln(1-x)}\right)$

(c) $\lim_{x\to 0} \dfrac{x(1-\cos x)}{1-\sqrt{1-x^3}}$

(d) $\lim_{x\to 0} \dfrac{\tan^3(3x)}{x^2 \sin x}$

Solution

(a) Note $e^{2x} = 1 + 2x + 2x^2 + \dots$
and $\ln(1+x) = x - \frac{1}{2}x^2 + \dots$
Then

$$\begin{aligned}\lim_{x\to 0} \frac{e^{2x}-1}{\ln(1+x)} &= \lim_{x\to 0} \frac{1+2x+2x^2+\dots-1}{x-\frac{1}{2}x^2+\dots} \\ &= \lim_{x\to 0} \frac{2x+2x^2+\dots}{x-\frac{1}{2}x^2+\dots} = \lim_{x\to 0} \frac{2+2x+\dots}{1-\frac{1}{2}x+\dots} \\ &= 2\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to 0} \left(\frac{1}{\ln(1+x)} + \frac{1}{\ln(1-x)}\right) &= \lim_{x\to 0} \frac{\ln(1-x)+\ln(1+x)}{\ln(1+x)\ln(1-x)} \\ &= \lim_{x\to 0} \frac{\ln(1-x^2)}{\ln(1+x)\ln(1-x)}\end{aligned}$$

$$= \lim_{x\to 0} \frac{-x^2-\frac{x^4}{2}+\dots}{(x-\frac{x^2}{2}+\frac{x^3}{3}-\dots)(-x-\frac{x^2}{2}-\frac{x^3}{3}+\dots)} = \lim_{x\to 0} \frac{-x^2-\frac{x^4}{2}+\dots}{-x^2-\frac{5}{12}x^4+\dots}$$

$$= \lim_{x\to 0} \frac{-1-\frac{x^2}{2}+\dots}{-1-\frac{5}{12}x^2+\dots} = 1$$

(c) Note

$$\begin{aligned}1-\cos x &= 1-(1-\frac{1}{2}x^2+\frac{1}{4!}x^4+\dots) \\ &= \frac{1}{2}x^2-\frac{1}{4!}x^4+\dots\end{aligned}$$

So

$$\begin{aligned}\lim_{x\to 0} \frac{x(1-\cos x)}{1-\sqrt{1-x^3}} &= \lim_{x\to 0} \frac{x(1-\cos x)(1+\sqrt{1-x^3})}{1-(1-x^3)} \\ &= \lim_{x\to 0} \frac{x(1-\cos x)(1+\sqrt{1-x^3})}{x^3}\end{aligned}$$

Note

$$\lim_{x\to 0}(1+\sqrt{1-x^3}) = 2,$$

$$\text{and} \lim_{x\to 0}\frac{x(1-\cos x)}{x^3} = \lim_{x\to 0}\frac{\frac{1}{2}x^3 - \frac{1}{4!}x^5 + \dots}{x^3} = \lim_{x\to 0}\frac{\frac{1}{2} - \frac{1}{4!}x^2 + \dots}{1} = \frac{1}{2}.$$

$$\implies \lim_{x\to 0}\frac{x(1-\cos x)}{1-\sqrt{1-x^3}} = 2\cdot\frac{1}{2} = 1$$

(d) Note

$$\tan x = x + \frac{1}{3}x^3 + \dots$$
$$\tan(3x) = 3x + 9x^3 + \dots$$
$$\sin x = x - \frac{1}{3!}x^3 + \dots$$

So

$$\lim_{x\to 0}\frac{\tan^3(3x)}{x^2\sin x} = \lim_{x\to 0}\frac{(3x+9x^3+\cdots)^3}{x^2(x-\frac{1}{6}x^3+\dots)}$$
$$= \lim_{x\to 0}\frac{27x^3+243x^5+\dots}{x^3-\frac{1}{6}x^5+\dots} = \lim_{x\to 0}\frac{27+243x^2+\dots}{1-\frac{1}{6}x^2+\dots}$$
$$= 27$$