THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics 2020-2021 Term 1
Homework Assignment 2

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1. The function f is continuous at $x = 0$ and is defined for $-1 < x < 1$ by

$$f(x) = \begin{cases} \frac{2a}{x}\ln(1+x) & \text{if } -1 < x < 0 \\ b & \text{if } x = 0 \\ \frac{x^2\cos x}{1-\sqrt{1-x^2}} & \text{if } 0 < x < 1. \end{cases}$$

Determine the values of the constants a and b.

Solution

For f to be continuous at $x = 0$,

(a) $\lim\limits_{x\to 0+} f(x) = f(0)$

$$\lim_{x\to 0+} \frac{x^2\cos x}{1-\sqrt{1-x^2}}$$
$$= \lim_{x\to 0+} \frac{x^2\cos x(1+\sqrt{1-x^2})}{1-(1-x^2)}$$
$$= \lim_{x\to 0+} \cos x(1+\sqrt{1-x^2})$$
$$= 2$$

So $b = 2$.

(b) $\lim\limits_{x\to 0-} f(x) = f(0)$

$$\lim_{x\to 0-} \frac{2a}{x}\ln(1+x)$$
$$= \lim_{y\to 0-} \frac{2a}{e^y-1}y \text{ (sub } 1+x = e^y)$$
$$= 2a$$
$$= 2$$

So $a = 1$.

2. Determine whether the following functions are differentiable at $x = 0$.

(a) $f(x) = \begin{cases} 5-2x, & \text{when } x < 0 \\ x^2-2x+5, & \text{when } x \geq 0 \end{cases}$

(b) $f(x) = \begin{cases} 1+3x-x^2, & \text{when } x < 0 \\ x^2+3x+2, & \text{when } x \geq 0 \end{cases}$

(c) $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$

(d) $f(x) = |\sin x|$

(e) $f(x) = x|x|$

Solution

(a)

$$\begin{aligned}\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0^+}\frac{x^2-2x+5-5}{x}\\ &= \lim_{x\to 0^+} x-2\\ &= -2\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0^-}\frac{5-2x-5}{x-0}\\ &= -2\end{aligned}$$

Hence, f is differentiable at $x=0$.
(b) Note that

$$\begin{aligned}\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} x^2+3x+2\\ &= 2\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} 1+3x-x^2\\ &= \lim_{x\to 0^-} 1 \neq 2\end{aligned}$$

Hence, f is not continuous at $x=0$, thus not differentiable at $x=0$.
(c)

$$\begin{aligned}\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0^+}\frac{e^{-\frac{1}{x^2}}}{x}\\ &= \lim_{y\to\infty} ye^{-y^2} \quad (\text{Let } y=\frac{1}{x})\\ &= \lim_{y\to\infty}\frac{y}{e^{y^2}}\\ &= \lim_{y\to\infty}\frac{1}{2ye^{y^2}} \quad (\text{L'Hopital})\\ &= 0\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0^-}\frac{e^{-\frac{1}{x^2}}}{x}\\ &= \lim_{y\to-\infty} ye^{-y^2} \quad (\text{Let } y=\frac{1}{x})\\ &= \lim_{y\to-\infty}\frac{y}{e^{y^2}}\\ &= \lim_{y\to-\infty}\frac{1}{2ye^{y^2}} \quad (\text{L'Hopital})\\ &= 0\end{aligned}$$

Hence, f is differentiable at $x = 0$.
(d)

$$\begin{aligned}\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0^+}\frac{|\sin x|-0}{x}\\ &= \lim_{x\to 0^+}\frac{\sin x}{x}\\ &= 1\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0^-}\frac{|\sin x|-0}{x}\\ &= \lim_{x\to 0^-}\frac{-\sin x}{x}\\ &= -1 \neq 1\end{aligned}$$

Hence, f is not differentiable at $x = 0$.
(e)

$$\begin{aligned}\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0^+}\frac{x|x|-0}{x}\\ &= \lim_{x\to 0^+}\frac{x^2}{x}\\ &= 0\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0^-}\frac{x|x|-0}{x}\\ &= \lim_{x\to 0^-}\frac{-x^2}{x}\\ &= 0\end{aligned}$$

3. Let $f(x) = |x|^3$.

   (a) Find $f'(x)$ for $x \neq 0$.

   (b) Show that $f(x)$ is differentiable at $x = 0$.

   (c) Determine whether $f'(x)$ is differentiable at $x = 0$.

   Solution (a)

$$f'(x) = \begin{cases} 3x^2, & \text{when } x > 0;\\ -3x^2, & \text{when } x < 0.\end{cases}$$

(b) Note that

$$\lim_{h\to 0}\frac{|h|^3-0}{h-0} = \lim_{h\to 0}\frac{|h|h^2}{h} = \lim_{h\to 0}|h|h = 0.$$

Hence f is differentiable at $x = 0$ with $f'(x) = 0$.

(c) Note that, by (a) and (b),

$$\lim_{h\to 0^+} \frac{f'(h) - f'(0)}{h - 0} = \lim_{h\to 0^+} \frac{3h^2}{h} = \lim_{h\to 0^+} 3h = 0.$$
$$\lim_{h\to 0^-} \frac{f'(h) - f'(0)}{h - 0} = \lim_{h\to 0^-} \frac{-3h^2}{h} = \lim_{h\to 0^-} -3h = 0.$$

Hence $f'(x)$ is differentiable at $x = 0$ with $f''(0) = 0$.

4. Let

$$f(x) = \begin{cases} (x-2)^2 \sin\left(\dfrac{1}{x-2}\right), & \text{when } x \neq 2; \\ 0, & \text{when } x = 2. \end{cases}$$

(a) Is f continuous on $\mathbb{R}$?
(b) Is f differentiable on $\mathbb{R}$?
(c) Is f' continuous on $\mathbb{R}$?

Solution

(a) We only need to check whether f is continuous at $x = 2$.
Since,

$$\lim_{x\to 2} (x-2)^2 = 0,$$

and

$$-1 \leq \sin \frac{1}{x-2} \leq 1$$

we have,

$$\lim_{x\to 2} (x-2)^2 \sin \frac{1}{x-2} = 0 = f(2)$$

so f is continuous at $x = 2$, and f is continuous on $\mathbb{R}$.

(b) Similarly, we only need to check whether f is differentiable at $x = 2$. since$(x - 2)^2 \sin \dfrac{1}{x-2}$ is differentiable on $x \neq 2$.
By definition,

$$\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{h^2 \sin(\frac{1}{h})}{h} = \lim_{h\to 0} h \sin(\frac{1}{h})$$

Because $\left|\sin(\frac{1}{h})\right| \leq 1$, we have,

$$\lim_{h\to 0} h \sin(\frac{1}{h}) = 0$$

So, f is differentiable at $x = 2$, moreover $f'(2) = 0$. f is differentiable on $\mathbb{R}$.

(c) According to (b), we have

$$f'(x) = \begin{cases} 2(x-2)\sin\left(\frac{1}{x-2}\right) + (-1)\cos\left(\frac{1}{x-2}\right), & \text{when } x \neq 2; \\ 0, & \text{when } x = 2. \end{cases}$$

However, $\lim_{x\to 2} f'(x)$ doesn't exist, since

$$\lim_{x\to 2} \cos\left(\frac{1}{x-2}\right)$$

doesn't exist.
So f' is not continuous at $x = 2$.

5. Find natural domains of the following functions and differentiate them on their natural domains. You are not required to do so from first principles.

(a) $f(x) = \dfrac{\sin\, x}{1 + \cos\, x}$.

(b) $f(x) = (1 + \tan^2 x)\cos^2 x$.

(c) $f(x) = \ln(\ln(\ln x))$

(d) $f(x) = \ln|\sin\, x|$

Solution

(a)

$$\begin{aligned} 1 + \cos x &= 0 \\ \cos x &= -1 \\ x &= (2n-1)\pi, n \in \mathbb{Z} \end{aligned}$$

Therefore, the natural domain is $\mathbb{R} \setminus \{(2n-1)\pi : n \in \mathbb{Z}\}$.

$$\begin{aligned} f'(x) &= \frac{(1+\cos x)\cos x - \sin x(-\sin x)}{(1+\cos x)^2} \\ &= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} \\ &= \frac{\cos x + 1}{(1+\cos x)^2} \\ &= \frac{1}{1+\cos x} \end{aligned}$$

(b) $\tan x$ is well-defined on $\mathbb{R} \setminus \{\frac{(2n-1)\pi}{2} : n \in \mathbb{Z}\}$. Therefore, this is also the natural domain of f.
Note that $f(x) = (1 + \tan^2 x)\cos^2 x = \cos^2 x + \sin^2 x = 1$. Hence, $f'(x) = 0$.
(c)

$$\begin{aligned} \ln x &> 0 && (1)\\ x &> 1 && (2)\\ \ln(\ln x) &> 0 && (3)\\ \ln x &> 1 && (4)\\ x &> e && (5) \end{aligned}$$

By considering the intersection of the intervals above, the natural domain is given by (e, ∞).

$$\begin{aligned} f'(x) &= \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{1}{x} \\ &= \frac{1}{x \ln x \ln(\ln x)} \end{aligned}$$

(d)

$$\begin{aligned} |\sin x| &> 0 \\ \sin x &\neq 0 \\ x &\neq n\pi, n \in \mathbb{Z} \end{aligned}$$

Therefore, the natural domain of f is $\mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$. Note that $\frac{d}{dx}(\ln |x|) = \frac{1}{x}$ for $x \neq 0$. Therefore,

$$\begin{aligned} f'(x) &= \frac{1}{\sin x} \cdot \cos x \\ &= \cot x \end{aligned}$$

6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$f(x + y) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Suppose f is differentiable at $x = 0$, with $f'(0) = a$. Show that f is differentiable at every $x \in \mathbb{R}$, and find $f'(x)$ in terms of a and $f(x)$.

Solution
Let $x = y = 0$, we have

$$f(0) = [f(0)]^2.$$

Hence $f(0) = 0$ or 1.
Case 1: $f(0) = 0$.
Let $y = 0$, we have, for any $x \in \mathbb{R}$

$$f(x) = f(x)f(0) = 0.$$

So, $f(x) \equiv 0$ for all $x \in \mathbb{R}$. In this case, f is differentiable at every $x \in \mathbb{R}$, and $f'(x) \equiv 0$.
Case 2: $f(0) = 1$
Since f is differentiable at $x = 0$, we have

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h) - 1}{h} = a.$$

Now we show f is differentiable for all x.
By definition,

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{f(h)f(x) - f(x)}{h} \\
&= f(x) \lim_{h\to 0} \frac{f(h) - 1}{h} \\
&= f(x)f'(0) \\
&= af(x).
\end{aligned}$$

Hence, f is differentiable for all x, and $f'(x) = af(x)$.

7. Find $\dfrac{dy}{dx}$ if
   (a) $x^2 + y^2 = e^{xy}$
   (b) $x^3y + \sin xy^2 = 4$
   (c) $y = \tan^{-1}\sqrt{x}$
   (d) $y = 3^{\sin x}$
   (e) $y = x^{\ln x}$
   (f) $y = x^{x^x}$

Solution

Find $\dfrac{dy}{dx}$ if

(a) $x^2 + y^2 = e^{xy}$

$$2x + 2y\frac{dy}{dx} = \left(y + x\frac{dy}{dx}\right) e^{xy}$$

$$\frac{dy}{dx} = \frac{ye^{xy} - 2x}{2y - xe^{xy}}$$

(b) $x^3y + \sin xy^2 = 4$

$$3x^2y + x^3\frac{dy}{dx} + \left(y^2 + 2xy\frac{dy}{dx}\right) \cos xy^2 = 0$$

$$\frac{dy}{dx} = \frac{-3x^2y - y^2\cos xy^2}{x^3 + 2xy\cos xy^2}$$

(c) $y = \tan^{-1}\sqrt{x}$

$\tan y = \sqrt{x}$

$\sec^2 y \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$

$\dfrac{dy}{dx} = \dfrac{\cos^2 y}{2\sqrt{x}} = \dfrac{1}{2\sqrt{x}(1+x)}$

(d) $y = 3^{\sin x}$

$\dfrac{dy}{dx} = 3^{\sin x} \ln 3 \cos x$

(e) $y = x^{\ln x}$

$\ln y = (\ln x)^2$

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{2\ln x}{x}$

$\dfrac{dy}{dx} = \dfrac{2y\ln x}{x} = \dfrac{2x^{\ln x}\ln x}{x}$

(f) $y = x^{x^x}$

$\ln y = x^x \ln x$

$\ln\ln y = x\ln x + \ln\ln x$

$\dfrac{1}{y\ln y}\dfrac{dy}{dx} = \ln x + 1 + \dfrac{1}{x\ln x}$

$\dfrac{dy}{dx} = (y\ln y)\left(\ln x + 1 + \dfrac{1}{x\ln x}\right) = x^{x^x} \cdot x^x \ln x \left(\ln x + 1 + \dfrac{1}{x\ln x}\right)$

8. Find $\dfrac{d^2y}{dx^2}$ if

(a) $y = \ln\tan x$

(b) $y = \sin^{-1}\sqrt{1-x^2}$

(c) $x^2 + y^2 = 1$

Solution

(a)

$$\frac{dy}{dx} = \frac{1}{\tan x} \cdot \sec^2 x = \frac{\cos x}{\sin x} \cdot \frac{1}{\cos^2 x} = \frac{1}{\sin x \cos x} = \frac{2}{\sin 2x} = 2\csc(2x)$$

$$\frac{d^2y}{dx^2} = -4\csc(2x)\cot(2x)$$

(b)

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}} \cdot \frac{-2x}{2\sqrt{1-x^2}} = -\frac{x}{\sqrt{x^2-x^4}}$$

$$\frac{d^2y}{dx^2} = -\frac{\sqrt{x^2-x^4} - x \cdot \frac{2x-4x^3}{2\sqrt{x^2-x^4}}}{x^2-x^4} = -\frac{x^2-x^4-x(x-2x^3)}{(x^2-x^4)^{\frac{3}{2}}} = -\frac{x^4}{(x^2-x^4)^{\frac{3}{2}}}$$

(c)

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{d^2y}{dx^2} = -\frac{y - x\frac{dy}{dx}}{y^2} = -\frac{y - x(-\frac{x}{y})}{y^2} = -\frac{x^2 + y^2}{y^3}$$

9. Find the n-th derivative of the following functions for all positive integers n.

(a) $f(x) = (e^x + e^{-x})^2$, $x \in \mathbb{R}$

(b) $f(x) = \dfrac{1}{1 - x^2}$, $x \in (-1, 1)$

(c) $f(x) = \sin x \cos x$, $x \in \mathbb{R}$

(d) $f(x) = \cos^2 x$, $x \in \mathbb{R}$

(e) $f(x) = \dfrac{x^2}{e^x}$, $x \in \mathbb{R}$

Solution

(a) Simplify $f(x)$ first,

$$f(x) = (e^x + e^{-x})^2 = e^{2x} + 2 + e^{-2x}.$$

Hence,

$$f^{(n)}(x) = 2^n e^{2x} + (-2)^n e^{-2x}.$$

(b) Process the partial fraction for $f(x)$. Suppose

$$f(x) = \frac{A}{1 + x} + \frac{B}{1 - x},$$

where A, B is a constant, then we have

$$\frac{1}{1 - x^2} = \frac{(B - A)x + (B + A)}{1 - x^2},$$

by comparing the coefficients, we have

$$\begin{cases} B + A &= 1, \\ B - A &= 0. \end{cases}$$

Hence, $A = B = \frac{1}{2}$, and

$$f(x) = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right).$$

Therefore,

$$f^{(n)}(x) = \frac{1}{2}\left[(-1)^n \frac{n!}{(1+x)^{n+1}} + \frac{n!}{(1-x)^{n+1}}\right].$$

(c) By double angle formula,

$$f(x) = \sin x \cos x = \frac{1}{2}\sin 2x.$$

Hence,

$$f^{(n)}(x) = \begin{cases} 2^{n-1}\sin 2x & \text{if } n = 4k \text{ for some } k \in \mathbb{N}, \\ 2^{n-1}\cos 2x & \text{if } n = 4k+1 \text{ for some } k \in \mathbb{N}, \\ -2^{n-1}\sin 2x & \text{if } n = 4k+2 \text{ for some } k \in \mathbb{N}, \\ -2^{n-1}\cos 2x & \text{if } n = 4k+3 \text{ for some } k \in \mathbb{N}. \end{cases}$$

(d) By double angle formula,

$$f(x) = \cos^2 x = \frac{1}{2}(1 + \cos 2x).$$

Hence,

$$f^{(n)}(x) = \begin{cases} 2^{n-1}\cos 2x & \text{if } n = 4k \text{ for some } k \in \mathbb{N}, \\ -2^{n-1}\sin 2x & \text{if } n = 4k+1 \text{ for some } k \in \mathbb{N}, \\ -2^{n-1}\cos 2x & \text{if } n = 4k+2 \text{ for some } k \in \mathbb{N}, \\ 2^{n-1}\sin 2x & \text{if } n = 4k+3 \text{ for some } k \in \mathbb{N}. \end{cases}$$

(e) Note that

$$f(x) = \frac{x^2}{e^x} = x^2 e^{-x} = g(x)h(x)$$

where $g(x) = x^2$, $h(x) = e^{-x}$. Using Leibniz Rule (proved by mathematical induction and product rule),

$$f^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} g^{(k)}(x)\, h^{(n-k)}(x).$$

Note that $g'(x) = 2x$, $g''(x) = 2$ and $g^{(k)}(x) = 0$ for all $k \geq 3$. Hence,

$$\begin{aligned} f^{(n)}(x) &= \binom{n}{0} g(x)\, h^{(n)}(x) \;+\; \binom{n}{1} g'(x)\, h^{(n-1)}(x) \;+\; \binom{n}{2} g''(x)\, h^{(n-2)}(x) \\ &= (-1)^n x^2 e^{-x} \;+\; (-1)^{n+1} 2nxe^{-x} \;+\; (-1)^n n(n-1)e^{-x}. \end{aligned}$$

10. (a) If $x^y = y^x$, where $x, y > 0$, show that

$$\frac{dy}{dx} = \frac{xy\ln y - y^2}{xy\ln x - x^2}$$

(b) Using implicit or inverse differentiation, show that

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

for $x \in (-1, 1)$.

(c) Let $f(x) = \arctan|x|$ for $x \in \mathbb{R}$. Find all $x \in \mathbb{R}$ such that f is differentiable at x, and find $f'(x)$ for all such x.

Solution

(a) Take logarithm, and then differentiate both sides with respect to x:

$$\begin{aligned} \frac{d}{dx}(y\ln x) &= \frac{d}{dx}(x\ln y) \\ \frac{dy}{dx}(\ln x) + \frac{y}{x} &= \ln y + \frac{x}{y}\frac{dy}{dx} \\ \frac{dy}{dx}(xy\ln x) + y^2 &= xy\ln y + x^2\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{xy\ln y - y^2}{xy\ln x - x^2}. \end{aligned}$$

(b) Let $y = \arcsin x$. Then $x = \sin y$, for $y \in (-\pi/2, \pi/2)$.

$$\frac{dx}{dy} = \cos y = \sqrt{1-\sin^2 y}.$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

(c) Suppose $x > 0$. Let $y = \arctan x$. Then $x = \tan y$, for $y \in (0, \pi/2)$.

$$\begin{aligned} \frac{dx}{dy} &= \frac{1}{\cos^2 y} = \frac{\sin^2 y + \cos^2 y}{\cos^2 y} = 1 + \tan^2 y. \\ \frac{dy}{dx} &= \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}. \end{aligned}$$

Hence f is differentiable for $x > 0$ with $f'(x) = \frac{1}{1+x^2}$.

By similar arguments, we can prove that f is differentiable for $x < 0$ with $f'(x) = -\frac{1}{1+x^2}$.

Now we prove that f is not differentiable at $x = 0$. By inverse differentiation, we know that $g(x) = \arctan x$ is differentiable at $x = 0$ with $g'(0) = 1$.

Hence f is not differentiable at $x = 0$ by noting the following facts.

$$\begin{aligned} \lim_{h\to 0^+}\frac{f(h)-f(0)}{h-0} &= \lim_{h\to 0^+}\frac{\arctan h}{h} = g'(0) = 1. \\ \lim_{h\to 0^-}\frac{f(h)-f(0)}{h-0} &= \lim_{h\to 0^-}\frac{\arctan(-h)}{h} = \lim_{h\to 0^-}\frac{-\arctan h}{h} = -g'(0) = -1. \end{aligned}$$

11. The chain rule says

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x),$$

or equivalently,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where $y = f(u)$ and $u = g(x)$.

(a) Give examples to show

$$(f \circ g)''(x) \neq f''(g(x)) \cdot g''(x),$$

or equivalently,

$$\frac{d^2y}{dx^2} \neq \frac{d^2y}{du^2} \cdot \frac{d^2u}{dx^2},$$

where $\frac{d^2y}{dx^2}$ denotes the second derivative of $y = f(x)$.

(b) Prove

$$(f \circ g)''(x) = f''(g(x)) \cdot (g'(x))^2 + f'(g(x)) \cdot g''(x).$$

Solution

(a) Let $y = u^2$ and $u = x$.
Then $y = x^2$.
$\frac{dy}{dx} = 2x$
$\frac{d^2y}{dx^2} = 2$
$\frac{d^2u}{dx^2} = 0$
$\frac{d^2y}{du^2} \cdot \frac{d^2u}{dx^2} = 0$

(b) Prove

$$(f \circ g)''(x) = f''(g(x)) \cdot (g'(x))^2 + f'(g(x)) \cdot g''(x).$$

Solution

$y = f(u)$ and $u = g(x)$.
$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
$\frac{d}{dx}\frac{dy}{dx}$
$= \frac{d}{dx}\left(\frac{dy}{du} \cdot \frac{du}{dx}\right)$

$$= \frac{d}{dx}\left(\frac{dy}{du}\right) \cdot \frac{du}{dx} + \frac{dy}{du} \cdot \frac{d^2u}{dx^2}$$
$$= \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du} \cdot \frac{d^2u}{dx^2}$$

12. (a) Suppose $a, b > 0$ are constants, and

$$y = \frac{1}{ab} \arctan\left(\frac{b}{a} \tan x\right)$$

for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Express $\frac{dy}{dx}$ as a function of $\sin x$ and $\cos x$.

(b) Suppose $a, b > 0$ are constants, and

$$y = \ln\left|\frac{a + b\tan x}{a - b\tan x}\right|$$

for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \left\{\pm \arctan \frac{a}{b}\right\}$. Express $\frac{dy}{dx}$ as a function of $\sin x$ and $\cos x$.

Solution

(a)

$$\frac{dy}{dx} = \frac{1}{ab} \frac{1}{1 + (\frac{b}{a} \tan x)^2} \cdot \frac{b}{a} \sec^2 x = \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x}$$

(b) Note that

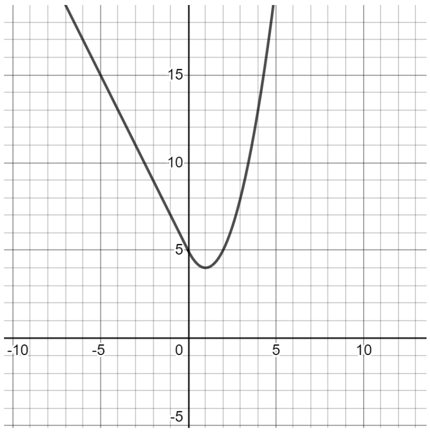
$$(\ln |x|)' = \frac{1}{x} \quad \text{for } x \neq 0$$

and

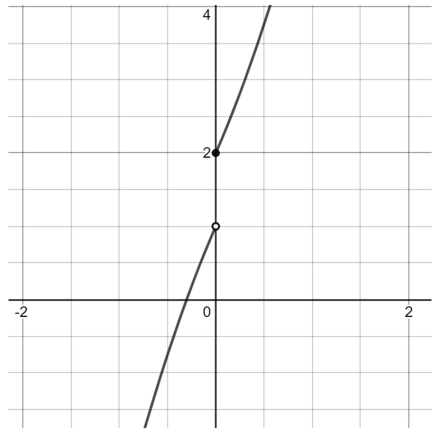
$$y = \ln(|\frac{a\cos x + b\sin x}{a\cos x - b\sin x}|) \quad \text{for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \left\{\pm \arctan \frac{b}{a}\right\}$$

Hence

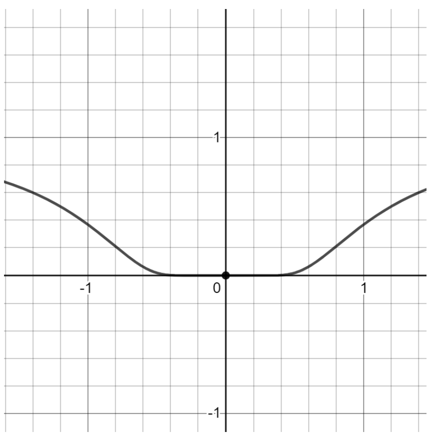
$$\frac{dy}{dx} = (\frac{a\cos x - b\sin x}{a\cos x + b\sin x})$$
$$\times (\frac{(a\cos x - b\sin x)(-a\sin x + b\cos x) - (a\cos x + b\sin x)(-a\sin x - b\cos x)}{(a\cos x - b\sin x)^2})$$
$$= \frac{2ab}{a\cos^2 x - b\sin^2 x}$$



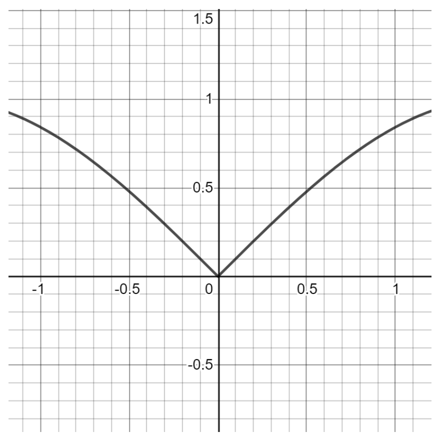
(a) 2a



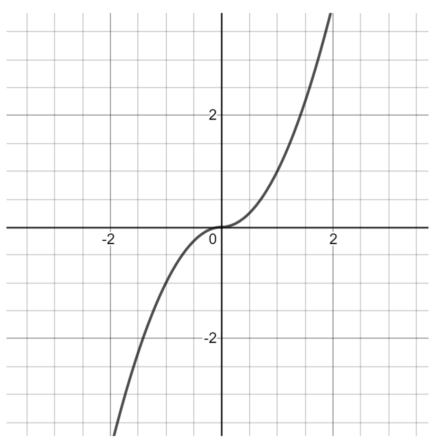
(b) 2b



(c) 2c



(d) 2d



(e) 2e

Figure 1: Graph of Q2

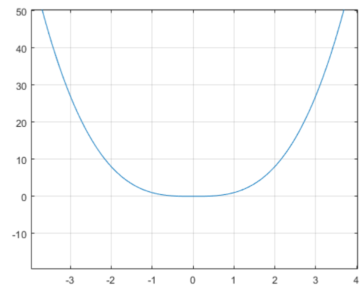


Figure 2: graph of f

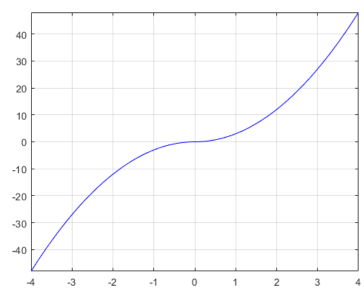


Figure 3: graph of f'

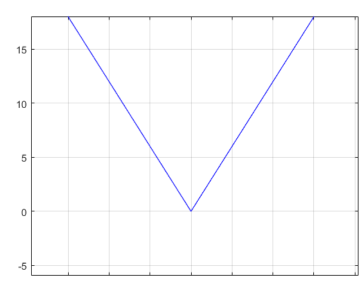
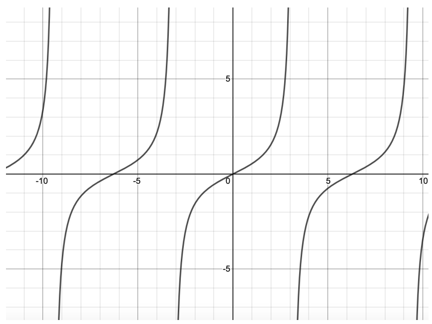
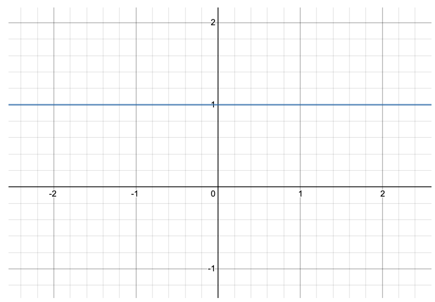


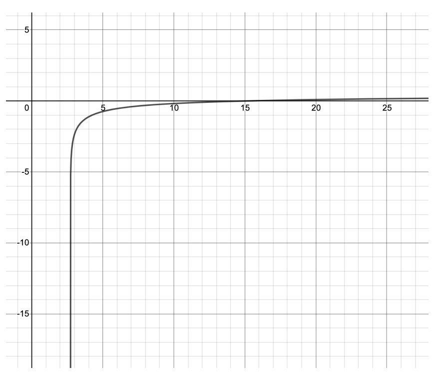
Figure 4: graph of f''



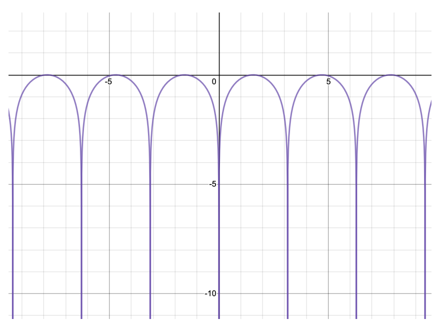
(a) 5a



(b) 5b



(c) 5c



(d) 5d

Figure 5: Graph of Q5