THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics **2020-2021 Term 1**
Homework Assignment 2
Due Date: 30 Oct 2020 (Friday)

General Regulations

- All assignments will be submitted and graded on Gradescope. You can view your grades and submit regrade requests here as well. For submitting your PDF homework on Gradescope, here are a few tips.

- Late assignments will receive a grade of 0.

- Print out the cover sheet (i.e. the first page of this document), and sign and date the statement of Academic Honesty.

- Write your COMPLETE name and student ID number legibly on the cover sheet (otherwise we will not take any responsibility for your assignments). Please write your answers using a black or blue pen, NOT any other color or a pencil.

- Write your solutions on A4 white paper. Please do not use any colored paper and make sure that your written solutions are a suitable size (easily read). Failure to comply with these instructions will result in a 10-point deduction).

- Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your answers getting good marks on this homework. Neatness and organization are also essential.

1. The function f is continuous at $x = 0$ and is defined for $-1 < x < 1$ by

$$f(x) = \begin{cases} \frac{2a}{x} \ln(1+x) & \text{if } -1 < x < 0 \\ b & \text{if } x = 0 \\ \frac{x^2 \cos x}{1 - \sqrt{1-x^2}} & \text{if } 0 < x < 1. \end{cases}$$

Determine the values of the constants a and b.

2. Determine whether the following functions are differentiable at $x = 0$.

(a) $f(x) = \begin{cases} 5 - 2x, & \text{when } x < 0 \\ x^2 - 2x + 5, & \text{when } x \geq 0 \end{cases}$

(b) $f(x) = \begin{cases} 1 + 3x - x^2, & \text{when } x < 0 \\ x^2 + 3x + 2, & \text{when } x \geq 0 \end{cases}$

(c) $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$

(d) $f(x) = |\sin x|$

(e) $f(x) = x|x|$

3. Let $f(x) = |x|^3$.

(a) Find $f'(x)$ for $x \neq 0$.

(b) Show that $f(x)$ is differentiable at $x = 0$.

(c) Determine whether $f'(x)$ is differentiable at $x = 0$.

4. Let

$$f(x) = \begin{cases} (x-2)^2 \sin\left(\frac{1}{x-2}\right), & \text{when } x \neq 2; \\ 0, & \text{when } x = 2. \end{cases}$$

(a) Is f continuous on $\mathbb{R}$?

(b) Is f differentiable on $\mathbb{R}$?

(c) Is f' continuous on $\mathbb{R}$?

5. Find natural domains of the following functions and differentiate them on their natural domains. You are not required to do so from first principles.

(a) $f(x) = \dfrac{\sin x}{1 + \cos x}$.

(b) $f(x) = (1 + \tan^2 x) \cos^2 x$.

(c) $f(x) = \ln(\ln(\ln x))$

(d) $f(x) = \ln|\sin x|$

6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$f(x + y) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Suppose f is differentiable at $x = 0$, with $f'(0) = a$. Show that f is differentiable at every $x \in \mathbb{R}$, and find $f'(x)$ in terms of a and $f(x)$.

7. Find $\dfrac{dy}{dx}$ if

(a) $x^2 + y^2 = e^{xy}$

(b) $x^3 y + \sin xy^2 = 4$

(c) $y = \tan^{-1} \sqrt{x}$

(d) $y = 3^{\sin x}$

(e) $y = x^{\ln x}$

(f) $y = x^{x^x}$

8. Find $\dfrac{d^2 y}{dx^2}$ if

(a) $y = \ln \tan x$

(b) $y = \sin^{-1} \sqrt{1 - x^2}$

(c) $x^2 + y^2 = 1$

9. Find the n-th derivative of the following functions for all positive integers n.

(a) $f(x) = (e^x + e^{-x})^2$, $x \in \mathbb{R}$

(b) $f(x) = \dfrac{1}{1 - x^2}$, $x \in (-1, 1)$

(c) $f(x) = \sin x \cos x$, $x \in \mathbb{R}$

(d) $f(x) = \cos^2 x$, $x \in \mathbb{R}$

(e) $f(x) = \dfrac{x^2}{e^x}$, $x \in \mathbb{R}$

10. (a) If $x^y = y^x$, where $x, y > 0$, show that

$$\frac{dy}{dx} = \frac{xy \ln y - y^2}{xy \ln x - x^2}$$

(b) Using implicit or inverse differentiation, show that

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1 - x^2}}$$

for $x \in (-1, 1)$.

(c) Let $f(x) = \arctan |x|$ for $x \in \mathbb{R}$. Find all $x \in \mathbb{R}$ such that f is differentiable at x, and find $f'(x)$ for all such x.

11. The chain rule says
$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x),$$
or equivalently,
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$
where $y = f(u)$ and $u = g(x)$.

(a) Give examples to show
$$(f \circ g)''(x) \neq f''(g(x)) \cdot g''(x),$$
or equivalently,
$$\frac{d^2y}{dx^2} \neq \frac{d^2y}{du^2} \cdot \frac{d^2u}{dx^2},$$
where $\dfrac{d^2y}{dx^2}$ denotes the second derivative of $y = f(x)$.

(b) Prove
$$(f \circ g)''(x) = f''(g(x)) \cdot (g'(x))^2 + f'(g(x)) \cdot g''(x).$$

12. (a) Suppose $a, b > 0$ are constants, and
$$y = \frac{1}{ab} \arctan\left(\frac{b}{a} \tan x\right)$$
for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Express $\dfrac{dy}{dx}$ as a function of $\sin x$ and $\cos x$.

(b) Suppose $a, b > 0$ are constants, and
$$y = \ln\left|\frac{a + b\tan x}{a - b\tan x}\right|$$
for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \left\{\pm \arctan \frac{a}{b}\right\}$. Express $\dfrac{dy}{dx}$ as a function of $\sin x$ and $\cos x$.