Math 1010C Term 1 2015
Supplementary exercises 7

1. (Putnam 2002) Let k be a fixed positive integer. The n-th derivative of $\dfrac{1}{x^k - 1}$ has the form $\dfrac{P_n(x)}{(x^k - 1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$ for all $n \geq 0$. (Hint: Use induction on n. More precisely, by differentiating $\dfrac{P_n(x)}{(x^k - 1)^{n+1}}$, find a recurrence relation between $P_{n+1}(1)$ and $P_n(1)$. Answer $= (-k)^n n!$.)

2. (Putnam 1998) Find the minimum value of
$$\frac{\left(x + \frac{1}{x}\right)^6 - \left(x^6 + \frac{1}{x^6}\right) - 2}{\left(x + \frac{1}{x}\right)^3 + \left(x^3 + \frac{1}{x^3}\right)}$$
for $x > 0$. (Hint: The substitution $t = x + \dfrac{1}{x}$ would work, but it is faster to observe that in the numerator, we have
$$x^6 + \frac{1}{x^6} + 2 = \left(x^3 + \frac{1}{x^3}\right)^2,$$
so that the numerator is just $A^2 - B^2$ if we write $A = \left(x + \frac{1}{x}\right)^3$ and $B = x^3 + \frac{1}{x^3}$.)

3. (Putnam 1998) Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that
$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$
(Hint: We would be done if any of the functions f, f', f'', f''' has a zero. So by continuity, we may assume that each of these four functions is either strictly positive, or strictly negative. Without loss of generality, we may assume $f'' > 0$ and $f''' > 0$ (why?). In this case, f' is strictly increasing and strictly convex, so $f'(x)$ must be positive for large enough x. As a result, f is strictly increasing and strictly convex on the half line $[b, \infty)$ when b is sufficiently large. Hence $f(x)$ must also be positive for large enough x.)