Math 1010C Term 1 2015
Supplementary exercises 2

1. Does the following limit exist? If yes, compute its value; if not, explain why not.
$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}-\cos\sqrt{x^2+1}}{\sqrt{x^2+1}+\cos\sqrt{x^2+1}}$$

2. (a) Find values of a and b such that
$$f(x) = \begin{cases} ax+2b, & x \le 0, \\ x^2+3a-b, & 0 < x \le 2, \\ 4x-2b, & x > 2 \end{cases}$$
is continuous at every $x \in \mathbb{R}$.

   (b) For such values of a and b, find all points on $\mathbb{R}$ at which f is differentiable.

3. (a) Suppose $f\colon [0,1] \to [0,1]$ is a continuous function on $[0,1]$. Show that there exists a point $x \in [0,1]$ such that $f(x) = x$.

   (b) Is the conclusion of part (a) still valid, if we replace the closed interval $[0,1]$ everywhere by the open interval $(0,1)$?

4. (a) Construct a (discontinuous) function $f\colon [0,1] \to \mathbb{R}$ that does not achieve a maximum on $[0,1]$.

   (b) Construct a continuous function $f\colon (0,1) \to \mathbb{R}$ that does not achieve a maximum on $(0,1)$.

   (c) Construct a continuous function $f\colon [0,\infty) \to \mathbb{R}$ that does not achieve a maximum on $[0,\infty)$. (Challenge: can you make this f bounded as well?)

   (d) The extreme value theorem says that if a function $f\colon [a,b] \to \mathbb{R}$ is continuous on a closed and bounded interval $[a,b]$, then there exists $c, d \in [a,b]$ such that $f(c) \le f(x) \le f(d)$ for all $x \in [a,b]$. Explain why the extreme value theorem does not apply in each of the above examples.