Math 1010C Term 1 2015
Supplementary exercises 1

1. A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *even* if $f(x) = f(-x)$ for all $x \in \mathbb{R}$, and *odd* if $f(x) = -f(-x)$ for all $x \in \mathbb{R}$.

   (a) Suppose $p\colon \mathbb{R} \to \mathbb{R}$ is the polynomial function

   $$p(x) = \sum_{n=0}^{d} a_n x^n.$$

   Show that p is even if and only if $a_n = 0$ for all odd integers n.

   (b) Let p be as in part (a). Find a necessary and sufficient condition on the coefficients of p, such that p is odd.

   (c) Is there a function $g\colon \mathbb{R} \to \mathbb{R}$ that is neither even nor odd?

   (d) Is there a function $h\colon \mathbb{R} \to \mathbb{R}$ that is both even and odd?

   (e) Show that every function $f\colon \mathbb{R} \to \mathbb{R}$ can be written as the sum of an odd function and an even function.

   (f) Show that the derivative of an odd function is even, and the derivative of an even function is odd.

   (g) For those who know some linear algebra: Does the set of all even functions from $\mathbb{R}$ to $\mathbb{R}$ form a vector space over $\mathbb{R}$? What about the set of all odd functions?

2. The following generalizes the concept of odd and even functions defined above.
   Suppose X is a set, and $\theta\colon X \to X$ is an *involution*, in the sense that $\theta \circ \theta$ is the identity function on X (i.e. $\theta(\theta(x)) = x$ for all $x \in X$).

   (a) Show that for any $y \in X$, there exists one and only one $x \in X$ such that $\theta(x) = y$. (We say that $\theta\colon X \to X$ is a *bijection*.)

   (b) A function $f\colon X \to \mathbb{R}$ is said to be even with respect to θ if $f(\theta(x)) = f(x)$ for all $x \in X$. A function $f\colon X \to \mathbb{R}$ is said to be odd with respect to θ if $f(\theta(x)) = -f(x)$ for all $x \in X$.

      (i) Find all functions $F\colon X \to \mathbb{R}$ that is both even with respect to θ, and odd with respect to θ.

(ii) Show that every function $f\colon X \to \mathbb{R}$ can be written as the sum $g+h$, where $g\colon X \to \mathbb{R}$ is odd with respect to θ, and $h\colon X \to \mathbb{R}$ is even with respect to θ.

(c) How is all this relevant to Question 1?

(d) For those who know complex numbers already: Did it matter that we considered functions that took values in $\mathbb{R}$? What if we considered complex-valued functions?

3. (a) Let L be a real number. Show that if $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} |f(x)| = |L|$. (Hint: Use $||f(x)| - |L|| \leq |f(x) - L|$.)

   (b) Show that the converse of the above implication is not true.

   (c) Show that the converse of the implication in (a) is true if $L = 0$. As a result,

$$\lim_{x\to a} f(x) = 0, \quad \text{if and only if} \quad \lim_{x\to a} |f(x)| = 0.$$

   (Hint: Use $-|f(x)| \leq f(x) \leq |f(x)|$.)

4. (a) Using the rules for computing limits, prove the product rule for derivatives. (Hint: Use

$$\frac{f(x)g(x) - f(c)g(c)}{x - c} = f(x)\frac{g(x) - g(c)}{x - c} + g(c)\frac{f(x) - f(c)}{x - c}).$$

   (b) Prove the quotient rule for derivatives.

5. Suppose $a_0 = \dfrac{10}{3}$ and $a_k = a_{k-1}^2 - 2$ for all $k \geq 1$.

   (a) Show that $a_k = 3^{2^k} + 3^{-2^k}$ for all $k \geq 0$.

   (b) Show that $\displaystyle\prod_{k=0}^{n} a_k = \frac{3^{2^{n+1}} - 3^{-2^{n+1}}}{3 - 3^{-1}}$ for all $n \geq 0$.

   (c) Show that $\displaystyle\prod_{k=0}^{n} (a_k - 1) = \frac{3^{2^{n+1}} + 3^{-2^{n+1}} + 1}{3 + 3^{-1} + 1}$ for all $n \geq 0$.

   (d) Compute $\displaystyle\lim_{n\to\infty} \prod_{k=0}^{n} \left(1 - \frac{1}{a_k}\right)$.