HYPERBOLIC FUNCTIONS

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In this note we *define* the hyperbolic sine and the hyperbolic cosine by the following power series. For any $x \in \mathbb{R}$, we define

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!},$$

and

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}.$$

This is possible since the radius of convergence of the two power series above are infinite. One should compare these formula to those defining the sine and the cosine: recall that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!},$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.$$

They are almost the same series, except that there are no minus signs at all in the series expansion of sinh and cosh.

Since the radius of convergence of the series defining sinh and cosh are infinite, we may thus differentiate term by term, and obtain that

$$\frac{d}{dx}\sinh x = \cosh x, \quad \text{and} \quad \frac{d}{dx}\cosh x = \sinh x.$$

(No minus sign in the last formula, as opposed to the derivative of cos!) Also, from this definition, it is clear that $\sinh 0 = 0$, $\cosh 0 = 1$, and $\sinh(-x) = -\sinh x$, $\cosh(-x) = \cosh x$ for all $x \in \mathbb{R}$.

Below we list some properties of sinh and cosh, along side with those of sin and cos. The reader is invited to provide proofs of all these properties (just follow what we have done for sin and cos).

$\cos^2 x + \sin^2 x = 1$	$\cosh^2 x - \sinh^2 x = 1$
$\sin(x+y) = \sin x \cos y + \cos x \sin y$	$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
$\cos(x+y) = \cos x \cos y - \sin x \sin y$	$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

Once we have the above compound angle formula, it is easy to derive the double angle formula:

$\sin(2x) = 2\sin x \cos x$	$\sinh(2x) = 2\sinh x \cosh x$
$\cos(2x) = \cos^2 x - \sin^2 x$	$\cosh(2x) = \cosh^2 x + \sinh^2 x$
$\cos(2x) = 2\cos^2 x - 1$	$\cosh(2x) = 2\cosh^2 x - 1$
$\cos(2x) = 1 - 2\sin^2 x$	$\cosh(2x) = 1 + 2\sinh^2 x$
$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$	$\cosh^2 x = \frac{1}{2}(1 + \cosh(2x))$
$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$	$\sinh^2 x = \frac{1}{2}(\cosh(2x) - 1)$

One may also derive the sum-to-product or product-to-sum formula. We leave this to the interested reader.

We close by mentioning the connection of all these to the exponential function. First, it is clear, from the definition of sinh and cosh, that

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

Indeed this allows us to derive all properties of sinh and cosh via the exponential function, rendering the memorization of most of the formulas above unnecessary.

On the other hand, if we introduce complex numbers, we can also express sin and cos in terms of the exponential function. First, the set of all complex numbers will be denoted by $\mathbb{C}$; it is the set of numbers of the form $a + bi$, where $i^2 = -1$, and $a, b \in \mathbb{R}$. They can be added, subtracted, multiplied and divided. Please refer to any standard text on basic properties of complex numbers.

To proceed further, one first shows that one can define a function $\exp\colon \mathbb{C} \to \mathbb{C}$ such that

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

for all complex numbers z (in particular, the series converges for all $z \in \mathbb{C}$). Then one verifies that

$$\exp(z)\exp(w) = \exp(z + w)$$

for all complex numbers $z, w \in \mathbb{C}$. Also, one checks that

$$\exp(ix) = \cos x + i \sin x$$

for all real numbers $x \in \mathbb{R}$. (This is the so-called Euler's identity.) It follows that for all $x \in \mathbb{R}$, we have

$$\cos x = \frac{1}{2}(\exp(ix) + \exp(-ix)) \tag{1}$$

and

$$\sin x = \frac{1}{2i}(\exp(ix) - \exp(-ix)). \tag{2}$$

Hence for any $x, y \in \mathbb{R}$, we have

$$\begin{aligned}
\sin(x + y) &= \frac{1}{2i}(\exp(i(x + y)) - \exp(-i(x + y))) \\
&= \frac{1}{2i}(\exp(ix)\exp(iy) - \exp(-ix)\exp(-iy)) \\
&= \frac{1}{2i}((\cos x + i \sin x)(\cos y + i \sin y) - (\cos x - i \sin x)(\cos y - i \sin y)) \\
&= \frac{1}{2i}(i \cos x \sin y + i \sin x \cos y + i \cos x \sin y + i \sin x \cos y) \\
&= \sin x \cos y + \cos x \sin y,
\end{aligned}$$

as in the table above. Similarly one can deduce the formula for $\cos(x+y)$. One can then deduce the double angle formula, the half-angle formula, etc as before.

In fact, sometimes one turns thing around, and define the sine and cosine of a complex number by formula (2) and (1): in other words, for $z \in \mathbb{C}$, sometimes people define

$$\sin z = \frac{1}{2i}(\exp(iz) - \exp(-iz))$$

and

$$\cos z = \frac{1}{2}(\exp(iz) + \exp(-iz)).$$

Then the compound angle formula continues to hold for this complex sine and cosine, by the same proof we just gave. They also admit the same power series expansions as in the real case:

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}$$

But they also have many new properties: the most notable one is that they are no longer bounded by 1 (in fact, one can check that $\cos(iy) = i\cosh y \to \infty$ as $y \to \infty$). You will learn more about these functions in complex analysis.