Math 1010A
Supplementary exercise about computations of Taylor polynomials

1. This problem provides a method that is often useful in computing Taylor polynomials of products / quotients / compositions of functions.
   (a) Let k be a positive integer. Suppose f is a function defined on an open interval I containing 0, and that f is k-times differentiable on I. Show that

$$\lim_{x\to 0} \frac{f(x) - \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} x^j}{x^k}$$

   exists, and is equal to $\dfrac{f^{(k)}(0)}{k!}$. (Hint: Apply L'Hopital's rule $(k-1)$ times, and then use the definition of $f^{(k)}(0)$.)
   (b) Let n be a non-negative integer. Suppose f is a function defined on an open interval I containing 0, and that f is n-times differentiable on I. Assume that there exists a polynomial P_n of degree $\leq n$, and a function E_n defined on I, such that

$$f(x) = P_n(x) + E_n(x) \quad \text{for all } x \in I, \text{ with} \quad \lim_{x\to 0} \frac{E_n(x)}{x^n} = 0.$$

   Show that
   (i) $\displaystyle\lim_{x\to 0} \frac{E_n(x)}{x^k} = 0$ for any non-negative integer $k \leq n$.
   (ii) $f^{(k)}(0) = P_n^{(k)}(0)$ for any non-negative integer $k \leq n$. (Hint: We proceed by induction on k. For $k = 0$, just recall $f(x) = P_n(x) + E_n(x)$, and let $x \to 0$. Assume now for some positive integer $k \leq n$, we have

$$\begin{cases} f(0) = P_n(0), \\ f'(0) = P_n'(0), \\ \quad\vdots \\ f^{(k-1)}(0) = P_n^{(k-1)}(0). \end{cases}$$

   We want to prove that $f^{(k)}(0) = P_n^{(k)}(0)$. But then by induction hypothesis,

$$\frac{f(x) - \sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} x^j}{x^k} = \frac{P_n(x) - \sum_{j=0}^{k-1} \frac{P_n^{(j)}(0)}{j!} x^j}{x^k} + \frac{E_n(x)}{x^k} \quad \text{for all } x \in I \setminus \{0\}.$$

   Since both f and P_n are k-times differentiable, letting $x \to 0$ and using (a), we get our desired conclusion.)
   (iii) P_n is the degree n Taylor polynomial of f centered at 0. (Hint: It suffices to show that for any polynomial P of degree n, we have

$$\sum_{k=0}^{n} \frac{P^{(k)}(0)}{k!} x^k = P(x).$$

   But if $P(x) = \displaystyle\sum_{k=0}^{n} a_k x^k$ for some coefficients $a_0, a_1, \ldots, a_n$, then differentiating both sides k times and setting $x = 0$, we get $a_k = \dfrac{P^{(k)}(0)}{k!}$ for any non-negative integer $k \leq n$. This concludes the proof.)

2. Below we see some applications of the earlier question to the computation of some Taylor polynomials.
   (a) The goal in this part is to compute the degree 23 Taylor polynomial of $\cosh(x^3)$ centered at 0.
      (i) Show that there exists a function A, defined on $\mathbb{R}$, such that

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + A(x) \quad \text{for all } x \in \mathbb{R}, \text{ with } \quad \lim_{x \to 0} \frac{A(x)}{x^\lambda} = 0 \quad \text{for any } \lambda < 8.$$

      (ii) Show that there exists a function B, defined on $\mathbb{R}$, such that

$$\cosh(x^3) = 1 + \frac{x^6}{2!} + \frac{x^{12}}{4!} + \frac{x^{18}}{6!} + B(x) \quad \text{for all } x \in \mathbb{R}, \text{ with } \quad \lim_{x \to 0} \frac{B(x)}{x^{23}} = 0.$$

      Hence find the degree 23 Taylor polynomial of $\cosh(x^3)$ centered at 0.
   (b) The goal in this part is to compute the degree 4 Taylor polynomial of $e^{-2x} \sin x$ centered at 0.
      (i) Show that there exists a function A, defined on $\mathbb{R}$, such that

$$e^{-2x} = 1 - 2x + 2x^2 - \frac{4x^3}{3} + A(x) \quad \text{for all } x \in \mathbb{R}, \text{ with } \quad \lim_{x \to 0} \frac{A(x)}{x^3} = 0.$$

      (ii) Show that there exists a function B, defined on $\mathbb{R}$, such that

$$\sin x = x - \frac{x^3}{6} + B(x) \quad \text{for all } x \in \mathbb{R}, \text{ with } \quad \lim_{x \to 0} \frac{B(x)}{x^4} = 0.$$

      (iii) Show that there exists a function C, defined on $\mathbb{R}$, such that

$$e^{-2x} \sin x = \left(1 - 2x + 2x^2 - \frac{4x^3}{3}\right)\left(x - \frac{x^3}{6}\right) + C(x) \quad \text{for all } x \in \mathbb{R}, \text{ with } \quad \lim_{x \to 0} \frac{C(x)}{x^4} = 0.$$

      Hence find the degree 4 Taylor polynomial of $e^{-2x} \sin x$ centered at 0.
   (c) The goal in this part is to compute the degree 5 Taylor polynomial of $\sec x = \dfrac{1}{\cos x}$ centered at 0.
      (i) Show that there exists a function A, defined on $\mathbb{R}$, such that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + A(x) \quad \text{for all } x \in \mathbb{R}, \text{ with } \quad \lim_{x \to 0} \frac{A(x)}{x^5} = 0.$$

      (ii) Show that there exists a function B, defined on $\mathbb{R}$, such that

$$\frac{1}{\cos x} = 1 + \left(\frac{x^2}{2!} - \frac{x^4}{4!}\right) + \left(\frac{x^2}{2!} - \frac{x^4}{4!}\right)^2 + B(x) \quad \text{for all } x \in \mathbb{R}, \text{ with } \quad \lim_{x \to 0} \frac{B(x)}{x^5} = 0.$$

      Hence find the degree 5 Taylor polynomial of $\sec x$ centered at 0.
   (d) Can you now combine the techniques in parts (b) and (c), to compute the degree 5 Taylor polynomial of $\tan x = \dfrac{\sin x}{\cos x}$ centered at 0?