MATH 1010A/K 2017-18

University Mathematics

Tutorial Notes III

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In this tutorial, you may use $\lim_{x\to+\infty} \frac{x^k}{e^x} = 0$ for any integer k and $\lim_{x\to 0} \frac{\sin x}{x} = 1$ without proof.

Question

(Q1) Evaluate the following limit.

(a) $\lim_{x\to 2} \frac{x^3-8}{x^4-2x-12}$.

(b) $\lim_{x\to 2} \frac{x+x^2+x^3+x^4-30}{x-2}$.

(c) $\lim_{x\to+\infty} \frac{\ln\left(e^{7x}+2x^9\right)}{\ln\left(e^{10x}+8x^7\right)}$

(d) $\lim_{x\to+\infty} \frac{\sqrt{3x^2+2}-1}{x}$

(e) $\lim_{x\to-\infty} \frac{\sqrt{3x^2+2}-1}{x}$

(Q2) Evaluate $\lim_{x\to\infty} \frac{\cos\cos x + \tan\sin x}{e^x}$.

(Q3) Evaluate $\lim_{x\to 0} \frac{\sin\left(x^3\right)}{x\tan^2 x}$.

(Q4) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \ln x + e^{-x+1} & \text{if } x \geq 1 \\ x^4 - 3x^2 + 2x & \text{if } x < 1 \end{cases}$$

(a) Compute $f'(x)$ for $x \neq 1$.

(b) Compute $\lim_{x\to 1^+} f'(x)$.

(c) Compute $\lim_{x\to 1^-} f'(x)$.

(d) Is f differentiable at 1? Justify your answer.

(Challenging Question) Suppose f, g are (real-valued) functions defined on an open interval containing 0, such that g is continuous at 0, and $g(0) \neq 0$.

For x sufficiently close to 0, define $u(x) = f(x)g(x)$, $v(x) = f(x)/g(x)$.

Suppose u and v are both differentiable at 0. Show that f is also differentiable at 0.

Answer

(A1)(a) Note $(2)^3 - 8 = 0 = (2)^4 - 2(2) - 12$, by root theorem, $x-2$ is a factor of x^3-8 and $x^4-2x-12$.

Using long division, $x^4 - 2x - 12 = (x-2)\left(x^3+2x^2+4x+6\right)$. Hence

$$\lim_{x\to 2}\frac{x^3-8}{x^4-2x-12} = \lim_{x\to 2}\frac{(x-2)\left(x^2+2x+4\right)}{(x-2)\left(x^3+2x^2+4x+6\right)} = \lim_{x\to 2}\frac{x^2+2x+4}{x^3+2x^2+4x+6} = \frac{12}{30} = \frac{2}{5}.$$

(b) One may use the method of long division, but here is another method.

$$\begin{aligned}
&\lim_{x\to 2}\frac{x+x^2+x^3+x^4-30}{x-2}\\
=&\lim_{x\to 2}\frac{(x-2)+\left(x^2-4\right)+\left(x^3-8\right)+\left(x^4-16\right)}{x-2}\\
=&\lim_{x\to 2}\frac{(x-2)+(x-2)(x+2)+(x-2)\left(x^2+2x+4\right)+(x-2)\left(x^3+2x^2+4x^3+8\right)}{x-2}\\
=&\lim_{x\to 2}\left[1+(x+2)+\left(x^2+2x+4\right)+\left(x^3+2x^2+4x^3+8\right)\right]\\
=&1+4+12+32=49
\end{aligned}$$

(c) Note that

$$\begin{aligned}
\lim_{x\to+\infty}\frac{\ln\left(e^{7x}+2x^9\right)}{\ln\left(e^{10x}+8x^7\right)} &= \lim_{x\to+\infty}\frac{\ln\left(1+\frac{2x^9}{e^{7x}}\right)+\ln\left(e^{7x}\right)}{\ln\left(1+\frac{8x^7}{e^{10x}}\right)+\ln\left(e^{10x}\right)}\\
&= \lim_{x\to+\infty}\frac{\ln\left(1+\frac{2x^9}{e^{7x}}\right)+7x}{\ln\left(1+\frac{8x^7}{e^{10x}}\right)+10x}\\
&= \lim_{x\to+\infty}\frac{\frac{1}{x}\cdot\ln\left(1+\frac{2x^9}{e^{7x}}\right)+7}{\frac{1}{x}\cdot\ln\left(1+\frac{8x^7}{e^{10x}}\right)+10}\\
&= \frac{0\cdot\ln(1+0)+7}{0\cdot\ln(1+0)+10}\\
&= \frac{7}{10}.
\end{aligned}$$

(d) $\displaystyle\lim_{x\to+\infty}\frac{\sqrt{3x^2+2}-1}{x} = \lim_{x\to+\infty}\frac{\sqrt{3x^2+2}}{\sqrt{x^2}} - \frac{1}{x} = \lim_{x\to+\infty}\left(\sqrt{3+\frac{2}{x^2}}-\frac{1}{x}\right) = \sqrt{3+0}-0 = \sqrt{3}.$

(e)

$$\lim_{x\to-\infty}\frac{\sqrt{3x^2+2}-1}{x} = \lim_{x\to-\infty}\frac{\sqrt{3x^2+2}}{-\sqrt{x^2}} - \frac{1}{x} = \lim_{x\to-\infty}\left(-\sqrt{3+\frac{2}{x^2}}-\frac{1}{x}\right) = -\sqrt{3+0}-0 = -\sqrt{3}.$$

(A2) Note that $\dfrac{-1-\tan 1}{e^x} \le \dfrac{\cos\cos x + \tan\sin x}{e^x} \le \dfrac{1+\tan 1}{e^x}$. (Check it!)

Note that $\displaystyle\lim_{x\to+\infty}\frac{-1-\tan 1}{e^x} = 0 = \lim_{x\to+\infty}\frac{1+\tan 1}{e^x}$,

by sandwich theorem, $\displaystyle\lim_{x\to+\infty}\frac{\cos\cos x+\tan\sin x}{e^x}$ exists and equal to 0.

(A3) $\displaystyle\lim_{x\to 0}\frac{\sin\left(x^3\right)}{x\tan^2 x} = \lim_{x\to 0}\left(\frac{\sin x^3}{x^3}\cdot\frac{x}{\sin x}\cdot\frac{x}{\sin x}\cdot\frac{\cos^2 x}{1}\right) = 1^3\cdot\frac{1}{1}\cdot\frac{1}{1}\cdot\frac{\cos^2 0}{1} = 1$

(A4) $f(x) = \begin{cases} \ln x + e^{-x+1} & \text{if } x \geq 1 \\ x^4 - 3x^2 + 2x & \text{if } x < 1 \end{cases}.$

(a) $f'(x) = \begin{cases} \dfrac{1}{x} - e^{-x+1} & \text{if } x > 1 \\ 4x^3 - 6x + 2 & \text{if } x < 1 \end{cases}.$

(b) $\lim_{x\to 1^+} f'(x) = \lim_{x\to 1^+} \left(\frac{1}{x} - e^{-x+1}\right) = 1 - e^{-1+1} = 0.$

(c) $\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^-} 4x^3 - 6x + 2 = 4 - 6 + 2 = 0.$

(d) Note that $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \left(\ln x + e^{-x+1}\right) = \ln 1 + e^0 = 1$

and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \left(x^4 - 3x^2 + 2x\right) = 1 - 3 + 2 = 0,$

these two limits are not equal, so $\lim_{x\to 1} f(x)$ does not exist,

that implies f is not continuous at $x = 1$.

Hence f is not differentiable at $x = 1$.

Remark:

It will be difficult for doing this by definition of differentiable since the function involving log, exp and polynomial of degree 4.