THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018-2019 semester 1 MATH4060
week 9 tutorial

Underlined contents were not included in the tutorial because of time constraint, but included here for completeness.

Various estimation skills were discussed and some propositions about holomorphic functions on the unit disc pertinent to the last question of the mid-term examination were discussed as well.

1 Elementary estimation

Determine whether each of the following statements is true.

1. For $\varepsilon > 0$, $\lim_{R\to+\infty} \int_R^\infty \frac{1}{1+x^{1+\varepsilon}} dx = 0$.
2. For $\varepsilon > 0$, $\lim_{R\to+\infty} \int_R^\infty \frac{1}{1+x^{\varepsilon}} dx = 0$.
3. Suppose $f(z) = \sum_{n\geq 1} a_n z^n$ converges on the unit disc and $a_1 \neq 0$. (Note that f has a simple zero at 0.) Then there exist positive constants c, C, δ such $c|z| \leq f(z) \leq C|z|$ for $|z| < \delta$.
4. Suppose f is entire. $|f(z)| \leq e^{|z|}$ for $|z| > 1$. Then there exists a positive constant A such that $|f(z)| \leq Ae^{|z|}$.
5. Suppose f is continuous and $\lim_{x\to+\infty} \frac{f(x)}{x} = 0$, then there exist a positive constants C such that for $x > 1$, $|f(x)| \leq Cx$.
6. Suppose $\lim_{x\to+\infty} \frac{f_n(x)}{x} = 0$. Then there exists a positive constant C such that whenever $x > 1$, $|f_n(x)| \leq C|x|$.
7. For every $\varepsilon > 0$, $\log x = O(x^\varepsilon)$ as $x \to \infty$.

(answer: T, F, T, T, T, F, T)

2 Holomorphic functions on the unit disc

Recall the last question of the mid-term examination asks for a proof of the following proposition.

Proposition 1. If a bounded holomorphic function f defined on the unit disc vanishes on $\{1 - 1/n : n \in \mathbb{N}\}$, then it is identically zero.

It can be proven by telescoping terms from $1-1/n$ and discarding other terms in Jensen's formula. The details can be found in the solution.

It is tempting to argue that the zeroes' accumulation at 1 is an obstruction. Such an approach is unlikely viable because the proposition hinges on the quantitative properties assumed, while an accumulation-of-zero argument is qualitative in nature. Counter-examples exist if various assumptions of the proposition are dropped, and indeed such functions cannot be extended beyond the unit circle at 1, and hence they do *not* violate isolation of zeros.

Facts below are mostly from Stein and Shakarchi's *Complex Analysis*. Example 2 is from Problem 1 of Chapter 2 and Propositions 4 and 5, Problem 1 and 2 of Chapter 5. Example 3 is inspired from Mr Lan's answer in the midterm.

Example 2. $f(z) = \sum_{n\geq 0} z^{2^n}$ is a holomorphic function on the unit disc that cannot be extended beyond the unit circle.

Example 3. $f(z) = \cos \frac{\pi}{2} \frac{z+1}{z-1}$ is a counter-example to Proposition 1 *without* the assumption of boundedess.

Remark. In fact, a counter-example exists as long as the prescribed zeros do not accumulate.

Proposition 4. Let $a_n \in \mathbb{D}$. Suppose

$$\sum (1 - |a_n|) < \infty. \tag{1}$$

Then there exists a bounded holomorphic function on the disc that vanishes exactly at a_n.

Proof. Consider $\varphi_a(z) = \frac{a-z}{1-\bar{a}_n z}$, which maps a to 0 and the unit disc to itself. Define $f(z) = \prod w_n \varphi_{a_n}(z)$, where $w_n \in \partial B(0,1)$ are to be chosen. Then if f converges locally uniformly, it is holomorphic, by Morera's theorem; bounded, since each factor maps into the unit disc; and vanishes exactly at $\{a_n\}$ because $\varphi_{a_n}(a_n) = 0$. For $|z| < r < 1$,

$$\begin{aligned}
\left| w_n \frac{a_n - z}{1 - \bar{a}_n z} - 1 \right| &= \left| \frac{w_n a_n - w_n z - 1 + \bar{a}_n z}{1 - \bar{a}_n z} \right| \\
&\leq \frac{|w_n a_n - 1| + |z||w_n - \bar{a}_n|}{|1 - \bar{a}_n z|} \\
&\leq \frac{|w_n||a_n - \bar{w}_n| + r|w_n - \bar{a}_n|}{1 - r} \\
&\leq \frac{1+r}{1-r} |w_n - \bar{a}_n|,
\end{aligned}$$

whose sum over n converges if it is chosen that $w_n = \bar{a}_n/|a_n|$ such that $|w_n - \bar{a}_n| = 1 - |a_n|$. Convergence then follows. ☐

The above result is optimal, as its converse holds.

Proposition 5. Suppose f is a bounded holomorphic function on the disc. Let $\{a_n\}$ be its set of zeros. Then (1) holds.

Proof. Dividing by z^m if necessary, it may be assumed that $f(0) \neq 0$. By Jensen's formula, for $0 < R < 1$, if $|a_n| \neq R$,

$$\sum -\chi_{|a_n|<R} \log \frac{|a_n|}{R} = \sum_{|a_n|<R} -\log \frac{|a_n|}{R} = \fint_{\partial B_R} \log|f| - \log|f(0)|.$$

Since RHS is bounded, by monotone convergence theorem / Fatou's lemma, $\sum -\log|a_n|$ converges. The result then follows from the fact that $-\log t \geq 1 - t$. □