MMAT 5011 Analysis II
Midterm Solution

1a. Let $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \geq 0 \right\}$

Note $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in S$, $(-1)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \notin S$

$\Rightarrow$ S is not a subspace

b. $\|e_1\|_{\frac{1}{2}} = \|(1,0)\|_{\frac{1}{2}} = (\sqrt{1} + \sqrt{0})^2 = 1$

$\|e_2\|_{\frac{1}{2}} = \|(0,1)\|_{\frac{1}{2}} = (\sqrt{0} + \sqrt{1})^2 = 1$

$\|e_1 + e_2\|_{\frac{1}{2}} = \|(1,1)\|_{\frac{1}{2}} = (\sqrt{1} + \sqrt{1})^2 = 4 > \|e_1\|_{\frac{1}{2}} + \|e_2\|_{\frac{1}{2}}$

Violates triangle inequality $\Rightarrow \|\cdot\|_{\frac{1}{2}}$ is not a norm

c. Let $a = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\cdots)$

Note $\displaystyle\sum_{i=1}^{2^n} |a_i| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}$

$\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} \cdots + \frac{1}{2^n} + \cdots + \frac{1}{2^n}$

$= 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}$

$= 1 + \frac{n}{2} \to \infty$ as $n \to \infty$

$\Rightarrow a \notin \ell^1$

Also, $\sum_{i=1}^{n} |a_i|^2 = 1 + \sum_{i=2}^{n} \frac{1}{i^2}$

$$\leq 1 + \int_1 \frac{1}{x^2} dx$$

$$= 1 + \left[\frac{-1}{x}\right]_1^n$$

$$< 2$$

$\Rightarrow \sum_{i=1}^{\infty} |a_i|^2 \leq 2$, $a \in \ell^2$

$\therefore \ \ell^2 \not\subseteq \ell^1$

2. Note $\| T_n(x) - T_m(x) \| = \| (T_n - T_m)(x) \|$

$$\leq \| T_n - T_m \| \, \| x \|$$

(T_n) is Cauchy

$\Rightarrow \forall \varepsilon > 0, \exists N > 0$ such that

$$\| T_n - T_m \| \leq \frac{\varepsilon}{1 + \|x\|} \quad \forall n, m > N$$

$\Rightarrow \| T_n(x) - T_m(x) \| \leq \frac{\varepsilon}{1 + \|x\|} \|x\| < \varepsilon \quad \forall n, m > N$

$\Rightarrow (T_n(x))$ is a Cauchy sequence in Y

Y is Banach space $\Rightarrow Y$ is complete

$\Rightarrow (T_n(x))$ is convergent

3. $\forall \varepsilon > 0$, let $I_n = \left(n - \frac{\varepsilon}{2^{n+2}}, n + \frac{\varepsilon}{2^{n+2}}\right)$

Then $\mathbb{N} \subseteq \bigcup_{n=1}^{\infty} I_n$

Also, $\sum_{n=1}^{\infty} |I_n| = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2} < \varepsilon$

$\Rightarrow$ $\mathbb{N}$ is a measure zero set.

4a. Suppose $x_1, x_2 \in X$ and $T(x_1) = T(x_2)$

Then $T(x_1 - x_2) = T(x_1) - T(x_2) = 0$

$\Rightarrow x_1 - x_2 \in N(T) = \{0\}$

$\Rightarrow x_1 - x_2 = 0$,

$\Rightarrow x_1 = x_2$

$\therefore$ T is injective

b. Suppose $x \in N(T)$, then

$\|x\| = \|T(x)\| = \|0\| = 0$

$\Rightarrow x = 0$

$\therefore N(T) = \{0\}$

By (a), T is injective.

5a. $\|f\|_2 = \|1+x\|_2$

$$= \sqrt{\int_0^1 (1+x)^2 dx}$$

$$= \sqrt{\left[\frac{1}{3}(1+x)^3\right]_0^1}$$

$$= \sqrt{8/3 - 1/3}$$

$$= \sqrt{7/3}$$

b. Note $|x_n| = \frac{1}{1+2^n} \leq \frac{1}{1+2} = |x_1| \quad \forall n \geq 1$

$$\Rightarrow \|x\|_\infty = \sup_n \|x_n\| = |x_1| = \frac{1}{3}$$

c. Note $\frac{1}{3} + \frac{1}{\frac{3}{2}} = \frac{1}{3} + \frac{2}{3} = 1$

$\therefore S : \ell^3 \longrightarrow (\ell^{\frac{3}{2}})'$ defined by

$$S(\vec{a})(\vec{x}) = \sum_{n=1}^\infty a_n x_n$$

is an isometry.

Let $a_n = \frac{1}{2^n}$. Then $S(\vec{a}) = T$

$$\|T\| = \|\vec{a}\|_3 = \left(\sum_{n=1}^\infty |a_n|^3\right)^{\frac{1}{3}} = \left(\sum_{n=1}^\infty \frac{1}{8^n}\right)^{\frac{1}{3}}$$

$$= \left(\frac{\frac{1}{8}}{1-\frac{1}{8}}\right)^{\frac{1}{3}} = \left(\frac{1}{7}\right)^{\frac{1}{3}}$$

6a. $\|p_n\| = \int_0^1 |x^n|\,dx = \left[\frac{1}{n+1}x^{n+1}\right]_0^1 = \frac{1}{n+1}$

b. $T(p_n) = (x^n)' = n x^{n-1} = n\,p_{n-1}$

$\Rightarrow \dfrac{\|T(p_n)\|}{\|p_n\|} = \dfrac{\|n\,p_{n-1}\|}{\|p_n\|} = \dfrac{n\left(\frac{1}{n}\right)}{\frac{1}{n+1}} = n+1 \to \infty$ as $n \to \infty$

$\therefore$ T is unbounded

c. $\dim P_n(\mathbb{R}) = n+1 < \infty$

$\Rightarrow$ $T_n : P_n(\mathbb{R}) \to P(\mathbb{R})$ is bounded

d. $\|U\| = \sup\limits_{0 \le x \le 1} |3x^3 - 2| = 2$

e. $\forall x \in [0,1]$,

$|S(p)(x)| = \left|\int_0^x p(t)\,dt\right| \le \int_0^x |p(t)|\,dt \le \int_0^1 |p(t)|\,dt = \|p\|$

$\Rightarrow \|S(p)\| = \int_0^1 |S(p)(x)|\,dx \le \int_0^1 \|p\|\,dt = \|p\|$

$\Rightarrow$ S is bounded.