MMAT 5011 Analysis II
2016-17 Term 2
Midterm Examination
March 14, 2017

Full mark: 80

1. (15 marks) Explain why the following statements are false. Justify each answer briefly by a counter-example.

   (a) The subset
   $$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) : a, b, c, d \geq 0 \right\}$$
   is a subspace of the real vector space $M_{2\times 2}(\mathbb{R})$ of all real 2×2 matrices.

   (b) The function
   $$\|\mathbf{x}\|_{1/2} = \left(\sqrt{|x_1|} + \sqrt{|x_2|} \right)^2 \quad \text{for } \mathbf{x} = (x_1, x_2)$$
   defines a norm on the vector space $\mathbb{R}^2$.

   (c) Recall that for $1 \leq p < \infty$, the real l^p-space is defined by
   $$l^p = \left\{ \mathbf{x} = (x_1, x_2, \ldots) : x_i \in \mathbb{R} \text{ for each } i \,, \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}.$$
   The space l^2 is a subset of l^1.

2. (10 marks) Let X be a normed space and Y be a Banach space. Suppose that (T_n) is a Cauchy sequence in $B(X, Y)$. Show that for any $x \in X$, $(T_n(x))$ is a convergent sequence in Y.

3. (10 marks) Show that the subset $\mathbb{N} = \{1, 2, 3, \ldots\}$ of $\mathbb{R}$ is measure zero.

4. (10 marks) Let X, Y be vector spaces and $T : X \to Y$ is a linear operator. Recall that the null space of T is defined to be the subspace
   $$N(T) = \{x \in X : T(x) = 0\}$$
   of X. Show that

   (a) If $N(T) = \{0\}$, then T is injective.

   (b) If $\|T(x)\| = \|x\|$ for any $x \in X$, then T is injective.

5. (15 marks) Find the following norms:

(a) $\|f\|_2$, where $f \in L^2([0,1])$ is defined by $f(x) = 1 + x$.

(b) $\|\mathbf{x}\|_\infty$, where $\mathbf{x} = (\frac{1}{1+2}, \frac{1}{1+2^2}, \ldots)$ is the sequence defined by $x_n = \dfrac{1}{1+2^n}$.

(c) $\|T\|$, where T is the linear functional on $l^{3/2}$ defined by

$$T(\mathbf{x}) = \sum_{n=1}^{\infty} \frac{x_n}{2^n}$$

for $\mathbf{x} = (x_1, x_2, \ldots) \in l^{3/2}$.

6. (20 marks) Let $P(\mathbb{R})$ be the normed space of real polynomials with norm given by

$$\|p\| = \int_0^1 |p(x)| dx$$

(a) Compute $\|p_n\|$, where $p_n(x) = x^n$.

(b) Show that differential operator $T : P(\mathbb{R}) \to P(\mathbb{R})$ defined by $T(p) = p'$ is unbounded.

(c) Let $P_n(\mathbb{R}) \subset P(\mathbb{R})$ be the subspace consisting of polynomials of degree at most n. Is the linear operator $T_n = T|_{P_n(\mathbb{R})}$, the restriction of T on $P_n(\mathbb{R})$, bounded? Explain briefly.

(d) Consider the bounded linear functional $U : P(\mathbb{R}) \to \mathbb{R}$ defined by

$$U(p) = \int_0^1 (3x^3 - 2)p(x) dx$$

Write down the norm of U. No justification is needed.

(e) Let $S : P(\mathbb{R}) \to P(\mathbb{R})$ be the linear operator defined by

$$S(p)(x) = \int_0^x p(t) dt.$$

Is S bounded? Explain briefly.