MMAT 5011 Analysis II

2016-17 Term 2

Assignment 3

Due date: Mar 7, 2017

Assume $\mathbb{F} = \mathbb{R}$ in all the following problems.

You do not have to turn in the solution of optional problems. However, you are encouraged to try all the problems.

1. Let X, Y be vector spaces and $T : X \to Y$ is a linear operator. Recall that the null space of T is defined to be the subspace

$$N(T) = \{x \in X : T(x) = 0\}$$

of X. Show that

   (a) If $N(T) = \{0\}$, then T is injective.

   (b) If $\|T(x)\| = \|x\|$ for any $x \in X$, then T is injective.

2. Let X be the normed space of real continuous functions on $[0, 1]$ with norm given by

$$\|f\|_1 = \int_0^1 |f(t)|dt.$$

Let $T : X \to \mathbb{R}$ be the linear functional defined by

$$T(f) = \int_0^1 f(t)(1 + 2t)dt.$$

   (a) Show that $\|T\| \le 3$;

   (b) Show that $\|T\| = 3$. (Hint: Consider monomials t^n.)

   (c) (Optional) Let $g(t)$ be a real continuous function on $[0, 1]$. Express the norm $\|S\|$ of the bounded linear functional $S : X \to \mathbb{R}$ defined by

$$S(f) = \int_0^1 f(t)g(t)dt$$

   in terms of g.

3. Let X be a normed space and Y be a Banach space. Suppose that (T_n) is a Cauchy sequence in $B(X, Y)$. Show that for any $x \in X$, $(T_n(x))$ is a convergent sequence in Y.

4. Consider the normed space $\mathbb{R}^2$ with standard Euclidean norm defined by $\|x\| = \sqrt{x_1^2 + x_2^2}$. Let $S : (\mathbb{R}^2)' \to \mathbb{R}^2$ be defined by

$$S(f) = (f(e_1), f(e_2)).$$

Show that S is norm-preserving, i.e. $\|S(f)\| = \|f\|$ for any $f \in (\mathbb{R}^2)'$.

5. Let V be a n-dimensional vector space and $f, g : V \to \mathbb{R}$ be non-zero linear functionals on V. Suppose that $N(f) = N(g) = Z$ is a $(n-1)$-dimensional subspace of V. Show that there exists a constant $c \in \mathbb{R}$ such that $f(x) = cg(x)$ for any $x \in V$. (Hint: Let $y \in V \setminus Z$. Note that every vector of V can be expressed uniquely as $z + \alpha y$ for some $z \in Z$ and $\alpha \in \mathbb{R}$.)

6. Let X be a normed space and $x, y \in X$. Suppose that $f(x) = f(y)$ for any bounded linear functional f on X. Show that $x = y$.

7. (Optional) Consider the following subspaces of the normed space l^∞:

$$Z = \{(x_1, x_2, \ldots) : \exists N > 0 \text{ such that } x_i = 0 \ \forall i \geq N\}$$
$$c = \left\{(x_1, x_2, \ldots) : \lim_{i\to\infty} x_i \text{ exists.}\right\}$$

Let $T : l^1 \to (l^\infty)'$ be defined by

$$T(y)(x) = \sum_{i=1}^{\infty} x_i y_i.$$

You may assume that T is well-defined and linear without proof.

(a) Consider the linear functional $f : c \to \mathbb{R}$ defined by $f(x) = \lim_{i\to\infty} x_i$. Show that f is bounded with $\|f\| = 1$.

(b) Suppose $y \in l^1$ and $T(y)(x) = 0$ for every $x \in Z$. Show that $y = 0$ and hence $T(y)$ is the zero linear functional on l^∞.

(c) Show that $\|T(y)\| = \|y\|$.

(d) Use the Hahn-Banach theorem and parts (a) and (b) to show that T is injective but not surjective.

8. (Optional) Read (or prove yourself!) section 2.10-7 on the dual space of l^p for $1 < p < \infty$.