Assignment 9.

(1) $\because$ M is a regular surface

$\therefore$ $\exists$ a coordinate charts $A = \{(X^{(k)}, U_k)\}$ covering M

[it is just a coordinate charts, we don't know if $X^{(k)} \circ (X^{(j)})^{-1}$ is orientation preserving]

$\because$ M has a continuous normal vector field $\vec{N}$

$\therefore$ For $(X^{(k)}, U_k)$

- if $$\frac{X_1^{(k)} \times X_2^{(k)}}{|X_1^{(k)} \times X_2^{(k)}|} = -N$$

then interchange u and v where $(u,v) \in U_k$

then $\tilde{X}_1^{(k)} = X_2^{(k)}$, $\tilde{X}_2^{(k)} = X_1^{(k)}$

$$\therefore \quad \frac{\tilde{X}_1^{(k)} \times \tilde{X}_2^{(k)}}{|\tilde{X}_1^{(k)} \times \tilde{X}_2^{(k)}|} = N$$

- if $$\frac{X_1^{(k)} \times X_2^{(k)}}{|X_1^{(k)} \times X_2^{(k)}|} = N$$

then $\tilde{X}_1^{(k)} = X_1^{(k)}$, $\tilde{X}_2^{(k)} = X_2^{(k)}$

$\therefore$ $\tilde{A} = \{(\tilde{X}^{(k)}, U_k)\}$ is a coordinate charts of M such that

$$\frac{\tilde{X}_1^{(k)} \times \tilde{X}_2^{(k)}}{|\tilde{X}_1^{(k)} \times \tilde{X}_2^{(k)}|} = N \quad \text{for } \forall\ (\tilde{X}^{(k)}, U_k)$$

$\therefore$ if $U_k \cap U_j \neq \phi$

then $$\frac{\tilde{X}_1^{(k)} \times \tilde{X}_2^{(k)}}{|\tilde{X}_1^{(k)} \times \tilde{X}_2^{(k)}|} = \frac{\tilde{X}_1^{(j)} \times \tilde{X}_2^{(j)}}{|\tilde{X}_1^{(j)} \times \tilde{X}_2^{(j)}|}$$

$\therefore$ $X^{(k)} \circ (X^{(j)})^{-1}$ is orientation preserving

$\therefore$ M is orientable.

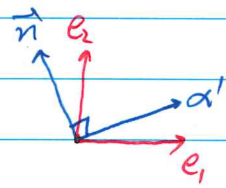
(2)(a): $\Gamma^1_{12}=\Gamma^1_{21}=\dfrac{\phi_v}{\phi}$, $\Gamma^2_{11}=-\dfrac{\phi\phi_v}{r^2}$, other Γ^k_{ij} are zero

$\therefore$ the geodesic equation is

$$u'' - \frac{2r\sin v}{a+r\cos v}u'v' = 0$$

$$v'' + \frac{(a+r\cos v)\sin v}{r}(u')^2 = 0$$

(b) See Tutorial 10.



(3) $\because$ α', $\vec{n}$ have the same orientation with e_1, e_2

$\alpha' = e_1\cos\theta + e_2\sin\theta$, $\quad |\vec{n}|=1$

$\therefore$ $\vec{n} = e_1\cos(\theta+\frac{\pi}{2}) + e_2\sin(\theta+\frac{\pi}{2})$

$= -e_1\sin\theta + e_2\cos\theta$

$\because$ $\alpha'' = e_1'\cos\theta + e_2'\sin\theta - e_1\sin\theta\cdot\theta' + e_2\cos\theta\cdot\theta'$

$\therefore$ $k_g = \langle \alpha'', \vec{n}\rangle$

$= \cos^2\theta\langle e_1', e_2\rangle - \sin^2\theta\langle e_1, e_2'\rangle + \sin^2\theta\cdot\theta' + \cos^2\theta\cdot\theta'$

$(\langle e_1', e_1\rangle = 0 = \langle e_2', e_2\rangle = \langle e_1, e_2\rangle)$

$= \cos^2\theta\langle e_1', e_2\rangle + \sin^2\theta\langle e_1', e_2\rangle + \theta'$

$(0 = \langle e_1, e_2\rangle' = \langle e_1', e_2\rangle + \langle e_1, e_2'\rangle)$

$= \langle e_1', e_2\rangle + \theta'$

$\because$ $g_{ij} = e^{2f}\delta_{ij}$

$\therefore$ $|X_1| = |X_2| = e^f$

$\therefore$ $e_1' = (e^{-f}X_1)' = -f'e^{-f}X_1 + e^{-f}(X_{11}u' + X_{12}v')$

$\therefore$ $k_g = e^{-2f}(u'\langle X_{11}, X_2\rangle + v'\langle X_{12}, X_2\rangle) + \theta'$

$= e^{-2f}(u'\langle h_{11}\vec{N} + \Gamma^k_{11}X_k, X_2\rangle + v'\langle h_{12}\vec{N} + \Gamma^k_{12}X_k, X_2\rangle) + \theta'$

$= e^{-2f}(u'e^{2f}\Gamma^2_{11} + v'e^{2f}\Gamma^2_{12}) + \theta'$ $\quad (\langle X_1, X_2\rangle = 0, \langle X_2, X_2\rangle = e^{2f})$

$= u'\Gamma^2_{11} + v'\Gamma^2_{12} + \theta'$

$\because\ \Gamma_{ij}^{k} = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$

$\therefore\ \Gamma_{11}^{2} = \frac{1}{2} g^{22} (\partial_1 g_{12} + \partial_1 g_{12} - \partial_2 g_{11})$ $(g^{21} = 0)$

$= \frac{1}{2} e^{-2f} (0 + 0 - 2f_2 e^{2f})$

$= -f_2$

$\Gamma_{12}^{2} = \frac{1}{2} g^{22} (\partial_1 g_{22} + \partial_2 g_{12} - \partial_2 g_{12})$ $(g^{21} = 0)$

$= \frac{1}{2} e^{-2f} (2f_1 e^{2f} + 0 - 0)$

$= f_1$

$\therefore\ k_g = \left(-u' \frac{\partial f}{\partial v} + v' \frac{\partial f}{\partial u}\right) + \theta'$