Assignment 8,

1, (a) Let $\alpha(\theta) = (a\cos\theta,\ b\sin\theta)$

$\therefore$ the curvature

$$k = \left| \frac{\det(\alpha', \alpha'')}{|\alpha'|^3} \right| = \frac{ab}{(a^2\sin^2\theta + b^2\cos^2\theta)^{\frac{3}{2}}}$$

$$\therefore k\Big|_{(a,0)} = k(0) = \frac{a}{b^2}$$

$$k\Big|_{(0,b)} = k(\tfrac{\pi}{2}) = \frac{b}{a^2}$$

(b) The plane is $(x,\ r\cos\theta_0,\ r\sin\theta_0)$ where $(x, r) \in R^2$, $\theta_0 \neq \frac{\pi}{2}$

$$\because \begin{cases} x^2 + y^2 = 1 \\ (x,y,z) \in (x,\ r\cos\theta_0,\ r\sin\theta_0) \end{cases} \Rightarrow x^2 + r^2\cos^2\theta_0 = 1$$

$\therefore$ it is an ellipse

and $\alpha(t) = (\cos t,\ \sin t,\ \sin t \cdot \tan\theta_0)$, $\alpha'(t) = (-\sin t,\ \cos t,\ \cos t \cdot \tan\theta_0)$

$\therefore \alpha''(t) = -(\cos t,\ \sin t,\ \sin t \cdot \tan\theta_0)$

- the geodesic curvature of α w.r.t the plane.

$\because \alpha''(t) \in$ the plane

$$\therefore |k_{g,\,plane}| = \left|\frac{\alpha' \times \alpha''}{|\alpha'|^3}\right|$$

$$\therefore |k_{g,\,plane}(0)| =$$

$$|k_{g,\,plane}(\tfrac{\pi}{2})| = \frac{1}{\cos\theta_0}$$

- the geodesic curvature of α w.r.t the cylinder.

$\alpha'' = k_g \vec{n} + k_n \vec{N}$

where $\vec{N} = (\cos t,\ \sin t,\ 0)$

$\therefore k_n = \langle \alpha'', \vec{N} \rangle = -1$

$\therefore |k_g| = |\alpha'' - k_n\vec{N}| = |\alpha'' + \vec{N}| = |(0, 0, -\sin t \cdot \tan\theta_0)| = |\sin t \cdot \tan\theta_0|$

$\therefore |k_{g,\,cylinder}(0)| = 0$

$|k_{g,\,cylinder}(\tfrac{\pi}{2})| = |\tan\theta_0|$

(2) $\alpha(t) = X(u^1(t), u^2(t))$

$\therefore \alpha' = X_i \dot{u}^i$ (where $X_i \dot{u}^i$ means $\sum_{i=1}^{2} X_i \dot{u}^i$)

$\therefore \alpha'' = (X_{ij} \dot{u}^j) \dot{u}^i + X_i \cdot \ddot{u}^i$

$= (h_{ij} N + \Gamma_{ij}^k X_k) \dot{u}^i \dot{u}^j + X_i \ddot{u}^i$

$\therefore (\alpha'')^T = \Gamma_{ij}^k X_k \dot{u}^i \dot{u}^j + X_i \ddot{u}^i$

$= (\Gamma_{ij}^k \dot{u}^i \dot{u}^j + \ddot{u}^k) X_k$

$\therefore (\alpha'')^T = 0$

$\Leftrightarrow \Gamma_{ij}^k \dot{u}^i \dot{u}^j + \ddot{u}^k = 0$ for $k = 1, 2$

(3) $\because \alpha'' \neq 0$

$\therefore$ We have the Frenet formula.

$$\begin{bmatrix} T_\alpha \\ N_\alpha \\ B_\alpha \end{bmatrix}' = \begin{bmatrix} 0 & k_\alpha & 0 \\ -k_\alpha & 0 & \tau_\alpha \\ 0 & -\tau_\alpha & 0 \end{bmatrix} \begin{bmatrix} T_\alpha \\ N_\alpha \\ B_\alpha \end{bmatrix} = \begin{bmatrix} 0 & k_\alpha & 0 \\ -k_\alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_\alpha \\ N_\alpha \\ B_\alpha \end{bmatrix}$$

since α is contained in a plane.

$\because \alpha$ lies on a regular surface and α is a geodesic

$\therefore \alpha'' = 0 + k_n \cdot \vec{N}$ (we can assume $|\alpha'| = 1$)

$\therefore T_\alpha' = k_n \cdot \vec{N}$

$\because T_\alpha' = k_\alpha \vec{N_\alpha}$ by the Frenet formula

$\therefore$ By choosing the direction of $\vec{N}$

we have $\vec{N} = \vec{N_\alpha}$

$\therefore \frac{d}{dt} N(\alpha(t)) = \frac{d}{dt} N_\alpha(\alpha(t)) = N_\alpha' = -k_\alpha T_\alpha$ by the Frenet formula.

$= -k_\alpha \alpha'$

$\therefore S_p(\alpha') = k_\alpha \alpha'$

$\therefore \alpha'$ is a principal direction.

(4) See question 3 of Tutorial 9