Assignment 6.

(1)(a) $\because F=0$

$\therefore$
$$A=\begin{vmatrix} -\frac{1}{2}E_{vv}-\frac{1}{2}G_{uu} & \frac{1}{2}E_u & -\frac{1}{2}E_v \\ -\frac{1}{2}G_u & E & 0 \\ \frac{1}{2}G_v & 0 & G \end{vmatrix}$$
$$= \left(-\frac{1}{2}E_v\right)\left(-\frac{1}{2}EG_v\right) + G\left(-\frac{1}{2}EE_{vv}-\frac{1}{2}EG_{uu}+\frac{1}{4}E_uG_u\right)$$
$$=\left(\frac{1}{4}EE_vG_v - \frac{1}{2}EGE_{vv}\right)+\left(-\frac{1}{2}EGG_{uu}+\frac{1}{4}GE_uG_u\right)$$

$$B=\begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & 0 \\ \frac{1}{2}G_u & 0 & G \end{vmatrix}$$
$$=\frac{1}{2}G_u\left(-\frac{1}{2}EG_u\right)+G\left(-\frac{1}{4}E_v^2\right)$$
$$=-\frac{1}{4}EG_u^2-\frac{1}{4}GE_v^2$$

$$\therefore A-B=\left(\frac{1}{4}EE_vG_v-\frac{1}{2}EGE_{vv}+\frac{1}{4}GE_v^2\right)+\left(\frac{1}{4}GE_uG_u-\frac{1}{2}EGG_{uu}+\frac{1}{4}EG_u^2\right)$$
$$=-\frac{1}{2}\sqrt{EG}\left[E_{vv}\sqrt{EG}-\frac{E_v}{2\sqrt{EG}}(EG_v+E_vG)\right]$$
$$-\frac{1}{2}\sqrt{EG}\left[G_{uu}\sqrt{EG}-\frac{G_u}{2\sqrt{EG}}(EG_u+GE_u)\right]$$
$$=-\frac{(EG)^{\frac{3}{2}}}{2}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v+\left(\frac{G_u}{\sqrt{EG}}\right)_u\right]$$
$$\therefore K=\frac{A-B}{(EG-0)^2}=-\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v+\left(\frac{G_u}{\sqrt{EG}}\right)_u\right]$$

(b) $\because E=G,\ F=0$
$$\therefore K=-\frac{1}{2E}\left[\left(\frac{E_v}{E}\right)_v+\left(\frac{E_u}{E}\right)_u\right]$$
$$=-\frac{1}{2E}\left(\frac{E_{vv}E-E_v^2}{E^2}+\frac{E_{uu}E-E_u^2}{E^2}\right)=-\frac{1}{2E^2}(E_{uu}+E_{vv})+\frac{1}{2E^3}(E_u^2+E_v^2)$$
$$\Delta f=\frac{1}{2}\Delta \lg E=\frac{1}{2}\left[\left(\frac{E_u}{E}\right)_u+\left(\frac{E_v}{E}\right)_v\right]$$
$$\therefore K=-\frac{1}{E}\Delta f=-e^{-2f}\Delta f$$

(2) $X(u,v) = \langle f(v)\cos u,\ f(v)\sin u,\ g(v) \rangle$

$\therefore\ X_u = \langle -f\sin u,\ f\cos u,\ 0 \rangle$

$\quad\ X_v = \langle f'\cos u,\ f'\sin u,\ g' \rangle$

$$\therefore\ g = \begin{bmatrix} f^2 & 0 \\ 0 & (f')^2+(g')^2 \end{bmatrix}$$

$$\therefore\ g^{-1} = \begin{bmatrix} \frac{1}{f^2} & 0 \\ 0 & \frac{1}{(f')^2+(g')^2} \end{bmatrix}$$

$$\therefore \left\{ \begin{aligned} \Gamma^k_{11} &= \frac{1}{2} g^{kl} \left(g_{1l,1} + g_{1l,1} - g_{11,l} \right) \\ \Gamma^1_{11} &= \frac{1}{2} g^{11} \left(g_{11,1} + g_{11,1} - g_{11,1} \right) \quad (g^{12}=0) \\ &= \frac{1}{2} \cdot \frac{1}{f^2} \cdot 0 = 0 \quad (\partial_u f = 0) \\ \Gamma^2_{11} &= \frac{1}{2} g^{22} \left(\cancelto{0}{g_{12,1}} + \cancelto{0}{g_{12,1}} - g_{11,2} \right) \quad (g^{21}=0) \end{aligned} \right.$$

$$= \frac{1}{2[(f')^2+(g')^2]} (-2ff') = -\frac{ff'}{(f')^2+(g')^2}$$

$$\left\{ \begin{aligned} \Gamma^k_{12} &= \frac{1}{2} g^{kl} \left(g_{1l,2} + g_{2l,1} - \cancelto{0}{g_{12,l}} \right) \\ \Gamma^1_{12} &= \frac{1}{2} g^{11} \left(g_{11,2} + \cancelto{0}{g_{21,1}} \right) \quad (g^{12}=0) \\ &= \frac{1}{2f^2} 2ff' = \frac{f'}{f} \\ \Gamma^2_{12} &= \frac{1}{2} g^{22} \left(\cancelto{0}{g_{12,2}} + g_{22,1} \right) \quad (g^{21}=0) \\ &= \frac{1}{2} \cdot \frac{1}{(f')^2+(g')^2} \cdot 0 = 0 \quad (\partial_u f = \partial_u g = 0) \end{aligned} \right.$$

$$\Gamma^k_{22} = \frac{1}{2} g^{kl} \left(g_{2l,2} + g_{2l,2} - g_{22,l} \right)$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11} \left(\cancelto{0}{g_{21,2}} + \cancelto{0}{g_{21,2}} - g_{22,1} \right) \quad (g^{12} = 0)$$

$$= \frac{1}{2f^2} \cdot 0 = 0 \quad (\partial_u f = \partial_u g = 0)$$

$$\Gamma^2_{22} = \frac{1}{2} g^{22} \left(g_{22,2} + g_{22,2} - g_{22,2} \right) \quad (g^{21} = 0)$$

$$= \frac{1}{2[(f')^2 + (g')^2]} \cdot (2f'f'' + 2g'g'') = \frac{f'f'' + g'g''}{(f')^2 + (g')^2}$$

3\. (a) $X(u,v) = (u\cos v,\ u\sin v,\ \ln u)$

$X_u = \langle \cos v,\ \sin v,\ \frac{1}{u} \rangle$

$X_v = \langle -u\sin v,\ u\cos v,\ 0 \rangle$

$$\therefore g = \begin{bmatrix} 1+\frac{1}{u^2} & 0 \\ 0 & u^2 \end{bmatrix}$$

$$\therefore K = -\frac{1}{2\sqrt{u^2+1}} \left[\left(\frac{0}{\sqrt{u^2+1}} \right)_v + \left(\frac{2u}{\sqrt{u^2+1}} \right)_u \right]$$

$$= -\frac{1}{2\sqrt{u^2+1}} \cdot \frac{2\sqrt{u^2+1} - 2u \cdot \frac{u}{\sqrt{u^2+1}}}{u^2+1} = -\frac{1}{(u^2+1)^2}$$

(b) $Y(u,v) = \langle u\cos v,\ u\sin v,\ v \rangle$

$Y_u = \langle \cos v,\ \sin v,\ 0 \rangle$

$Y_v = \langle -u\sin v,\ u\cos v,\ 1 \rangle$

$$\therefore g = \begin{bmatrix} 1 & 0 \\ 0 & u^2+1 \end{bmatrix}$$

$$\therefore K = -\frac{1}{2\sqrt{u^2+1}} \left[\left(\frac{0}{\sqrt{u^2+1}} \right)_v + \left(\frac{2u}{\sqrt{u^2+1}} \right)_u \right] = -\frac{1}{(u^2+1)^2}$$

$\therefore K_X = K_Y \quad \forall (u,v) \in \mathbb{R}^2$

but $g_X \neq g_Y$