Assignment 5.

1. Let k_1, k_2 be the principle curvatures of S. $(k_1 \geq k_2)$

then $H = \frac{k_1+k_2}{2}$, $K = k_1 \cdot k_2$

$\therefore$ (a) $H^2 - K = \frac{k_1^2 + 2k_1k_2 + k_2^2}{4} - k_1k_2 = \frac{k_1^2 - 2k_1k_2 + k_2^2}{4} = \left(\frac{k_1 - k_2}{2}\right)^2 \geq 0$

(b) $H + \sqrt{H^2-K} = \frac{k_1+k_2}{2} + \frac{k_1-k_2}{2} = k_1$

$H - \sqrt{H^2-K} = \frac{k_1+k_2}{2} - \frac{k_1-k_2}{2} = k_2$

2. Let $\vec{V_1}, \vec{V_2}$ be the eigenvectors of S_p corresponding to eigenvalues k_1, k_2 and $\{\vec{V_1}, \vec{V_2}\}$ is an orthonormal basis of T_pS.

1° $\because T' = k_g \vec{n} + k_n \vec{N}$

$\therefore k_n = \langle T', \vec{N} \rangle = \langle \frac{d^2}{dt^2}\alpha(t), \vec{N} \rangle = \frac{d}{dt}\langle \alpha'(t), \vec{N} \rangle - \langle \alpha'(t), \frac{d}{dt}N(\alpha(t)) \rangle$

$= \langle \alpha'(t), S_p(\alpha'(t)) \rangle$ Since $\langle \alpha'(t), \vec{N} \rangle \equiv 0$.

2° $\forall\ \vec{V} \in T_pS$ with $|\vec{V}| = 1$ and $\vec{V}$ makes an angle θ with a fixed direction.

$\because \{\vec{V_1}, \vec{V_2}\}$ is an orthonormal basis of T_pS

$\therefore \vec{V} = \cos(\theta+\theta_0)\vec{V_1} + \sin(\theta+\theta_0)\vec{V_2}$ for a fixed θ_0

$\therefore S_p(\vec{V}) = \cos(\theta+\theta_0)k_1\vec{V_1} + \sin(\theta+\theta_0)k_2\vec{V_2}$

$\therefore k_n(\theta) = \langle \vec{V}, S_p(\vec{V}) \rangle$

$= \langle \cos(\theta+\theta_0)\vec{V_1} + \sin(\theta+\theta_0)\vec{V_2}, \cos(\theta+\theta_0)k_1\vec{V_1} + \sin(\theta+\theta_0)k_2\vec{V_2} \rangle$

$= \cos^2(\theta+\theta_0)k_1 + \sin^2(\theta+\theta_0)k_2$

3° $\frac{1}{2\pi}\int_0^{2\pi} k_n(\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi} \cos^2(\theta+\theta_0)k_1 + \sin^2(\theta+\theta_0)k_2\, d\theta$

$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1+\cos(2\theta+2\theta_0)}{2}k_1 + \frac{1-\cos(2\theta-\theta_0)}{2}k_2\, d\theta$

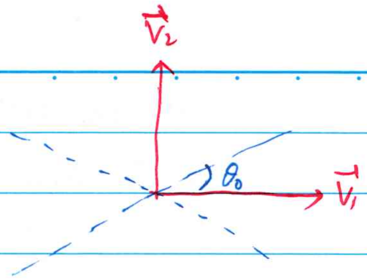
$= \frac{1}{2\pi}\left(\frac{k_1}{2}\cdot 2\pi + \frac{k_2}{2}\cdot 2\pi\right)$

$= \frac{k_1+k_2}{2} = H$.

3, $\because p$ is a hyperbolic point

$\therefore k_1 < 0,\ k_2 > 0$

let $\vec{v}$ be a asymptotic direction



then $\vec{v} = \cos\theta \vec{v_1} + \sin\theta \vec{v_2}$ for some θ

and $0 = k_n(\vec{v}) = \langle \vec{v}, S_p(\vec{v}) \rangle = \cos^2\theta\, k_1 + \sin^2\theta\, k_2$

$$\therefore \frac{\sin^2\theta}{\cos^2\theta} = -\frac{k_1}{k_2}$$

$\therefore \theta = -\theta_0, \theta_0, \pi-\theta_0, \theta_0-\pi$ for some $\theta_0 \in (0, \frac{\pi}{2})$

$\therefore$ The principle directions bisect the asymptotic direction.

4. $\because$ M is inside a sphere and M is tangent to $S(r)$ at P

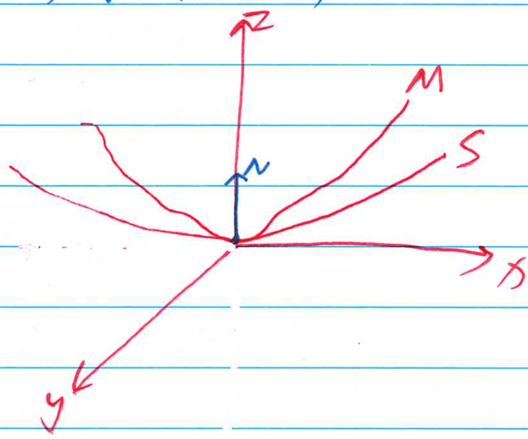
$\therefore$ After rotation and translation, any $\vec{V} \in T_pM$ with $|\vec{V}|=1$

we can assume $P=(0,0,0)$, $\vec{V}=(1,0,0)$, $\vec{N}_p=(0,0,1)$

$M=(x,y,G(x,y))$ around P

$S=(x,y,H(x,y))$ around P where $H(x,y)=r-\sqrt{r^2-(x^2+y^2)}$

and we have $G \geq H$ near P.



1° $\because G(0,0)=H(0,0)=0$

$G(x,y) \geq H(x,y) \geq 0$ near $(0,0)$

$\therefore \partial_x G(0,0) \geq \partial_x H(0,0)$

$\partial_x^2 G(0,0) \geq \partial_x^2 H(0,0) = \frac{1}{r}$

using $f''(x) = \lim_{t \to 0} \frac{f(x+t)+f(x-t)-2f(x)}{t^2}$

2°, $\partial_x M = (1,0,\partial_x G)$, $\vec{N}_M = (f,g,h)$ with $h_p = 1$

$\because \langle \partial_x M, \vec{N} \rangle \equiv 0$

$\therefore f + h \cdot \partial_x G = 0$

$\therefore f_x + h_x \partial_x G + h \partial_x^2 G = 0$

$\therefore \langle \partial_x M, \partial_x \vec{N} \rangle |_{(0,0)} = \langle \vec{V}, \partial_{\vec{V}} \vec{N} \rangle$

$= f_x + h_x \partial_x G |_{(0,0)}$

$= -h \partial_x^2 G |_{(0,0)}$

$= -\partial_x^2 G$

$\leq -\frac{1}{r}$

$\therefore \langle \vec{V}, S_p(\vec{V}) \rangle \geq \frac{1}{r}$ for $\forall \vec{V} \in T_pM$ and $|\vec{V}|=1$

$\therefore \begin{cases} k_1 \geq \frac{1}{r} \\ k_2 \geq \frac{1}{r} \end{cases}$

$\therefore K_M(P) \geq \frac{1}{r^2}$