Assignment 4.

1. • $\partial_u X = (1-u^2+v^2,\ 2uv,\ -2u)$

$\partial_v X = (2uv,\ 1-v^2+u^2,\ 2v)$

$\vec{N} = \dfrac{(\partial_u X)\times(\partial_v X)}{|(\partial_u X)\times(\partial_v X)|} = \dfrac{(2u,\ -2v,\ 1-u^2-v^2)}{1+u^2+v^2}$

$\partial_u\partial_u X = (-2u,\ 2v,\ -2)$

$\partial_v\partial_u X = \partial_u\partial_v X = (2v,\ 2u,\ 0)$

$\partial_v\partial_v X = (2u,\ -2v,\ 2)$

• $g = \begin{bmatrix} (1+u^2+v^2)^2 & 0 \\ 0 & (1+u^2+v^2)^2 \end{bmatrix}$

$h = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$

• $K(p) = \dfrac{\det(h)}{\det(g)} = \dfrac{-4}{(1+u^2+v^2)^4}$

$H(p) = \dfrac{1}{2}\cdot\dfrac{2(1+u^2+v^2)^2 - 2(1+u^2+v^2)^2}{(1+u^2+v^2)^4} = 0$

2. (a) $X_x = (1,0,2x)$, $X_y = (0,1,2ky)$

$\therefore\ T_pM = \text{span}\{(1,0,0),\ (0,1,0)\}$

(b) $X_{xx} = (0,0,2)$, $X_{xy} = (0,0,0)$, $X_{yy} = (0,0,2k)$

$\vec{N}_p = \dfrac{X_x \times X_y}{|X_x \times X_y|} = (0,0,1)$

$\therefore\ g(p) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $h(p) = \begin{bmatrix} 2 & 0 \\ 0 & 2k \end{bmatrix}$

$S_p = h\cdot g^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 2k \end{bmatrix}$

$K(p) = \dfrac{4k}{1} = 4k$

$H(p) = \dfrac{1}{2}\cdot\dfrac{2+2k}{1} = k+1$

3, the equation of the surface is

$$X(t,s) = (\sin t\cos s,\ \sin t \sin s,\ \cos t + \log \tan\tfrac{t}{2})$$

- $X_t = (\cos t\cos s,\ \cos t\sin s,\ -\sin t + \frac{1}{\sin t})$

  $X_s = (-\sin t\sin s,\ \sin t\cos s,\ 0)$

- $\vec{N} = \dfrac{X_t \times X_s}{|X_t \times X_s|} = (-\cos t\cos s,\ -\cos t\sin s,\ \sin t)$

- $X_{tt} = (-\sin t\cos s,\ -\sin t\sin s,\ -\cos t - \frac{\cos t}{\sin^2 t})$

  $X_{ts} = (-\cos t\sin s,\ \cos t\cos s,\ 0)$

  $X_{ss} = (-\sin t\cos s,\ -\sin t\sin s,\ 0)$

- $g = \begin{bmatrix} \frac{\cos^2 t}{\sin^2 t} & 0 \\ 0 & \sin^2 t \end{bmatrix}$

  $h = \begin{bmatrix} -\frac{\cos t}{\sin t} & 0 \\ 0 & \sin t\cos t \end{bmatrix}$

- $K(p) = \dfrac{\det(h)}{\det(g)} = \dfrac{-\cos^2 t}{\cos^2 t} = -1$

4, 1°, $d(f\vec{N})(\vec{v}) = \partial_{\vec{v}} f \cdot \vec{N} + f\,\partial_{\vec{v}}\vec{N}$

$\therefore\ d(f\vec{N})(\vec{v}_1) \times d(f\vec{N})(\vec{v}_2)$

$= [\partial_{v_1} f \cdot \vec{N} + f\partial_{v_1}\vec{N}] \times [\partial_{v_2} f\cdot\vec{N} + f\partial_{v_2}\vec{N}]$

$= -f\partial_{v_2} f(\vec{N}\times\partial_{v_1}\vec{N}) + f\partial_{v_1} f(\vec{N}\times\partial_{v_2}\vec{N}) + f^2(\partial_{v_1}\vec{N}\times\partial_{v_2}\vec{N})$

$\therefore\ \langle d(f\vec{N})(\vec{v}_1)\times d(f\vec{N})(\vec{v}_2), \vec{N}\rangle$

$= f^2\langle \partial_{v_1}\vec{N}\times\partial_{v_2}\vec{N}, \vec{N}\rangle$ since $\vec{N}\perp(\partial_{v_1}\vec{N}\times\vec{N})$ $\vec{N}\perp(\partial_{v_2}\vec{N}\times\vec{N})$

$\therefore\ \dfrac{\langle d(f\vec{N})(\vec{v}_1)\times d(f\vec{N})(\vec{v}_2), \vec{N}\rangle}{f^2} = \langle\partial_{v_1}\vec{N}\times\partial_{v_2}\vec{N}, \vec{N}\rangle$

2°, $\because\ \begin{bmatrix} S_p(\partial_u) \\ S_p(\partial_v)\end{bmatrix} = \begin{bmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{bmatrix}\begin{bmatrix}\partial_u \\ \partial_v\end{bmatrix} = h\cdot g^{-1}\begin{bmatrix}\partial_u \\ \partial_v\end{bmatrix}$

let $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d\end{bmatrix}\begin{bmatrix}\partial_u \\ \partial_v\end{bmatrix}$

$\therefore\ \begin{bmatrix}\partial_{v_1} N \\ \partial_{v_2} N\end{bmatrix} = \begin{bmatrix} -S_p(v_1) \\ -S_p(v_2)\end{bmatrix} = -\begin{bmatrix} a & b \\ c & d\end{bmatrix}\begin{bmatrix} S_p(\partial_u) \\ S_p(\partial_v)\end{bmatrix} = -\begin{bmatrix} a & b \\ c & d\end{bmatrix} h\cdot g^{-1}\begin{bmatrix}\partial_u \\ \partial_v\end{bmatrix}$

$\therefore \partial_{v_1} N \times \partial_{v_2} N$

$= \det\left(-\begin{bmatrix} a & b \\ c & d \end{bmatrix} h g^{-1}\right) \partial_u \times \partial_v$

$\because N = v_1 \times v_2$

$= (a\partial_u + b\partial_v) \times (c\partial_u + d\partial_v)$

$= (ad-bc)\, \partial_u \times \partial_v$

$\therefore \partial_{v_1} N \times \partial_{v_2} N$

$= (ad-bc) \cdot \det(h) \cdot \frac{1}{\det(g)} \cdot \frac{1}{ad-bc} N$

$= \frac{\det(h)}{\det(g)} N$

$\therefore \langle \partial_{v_1} N \times \partial_{v_2} N, N \rangle$

$= \frac{\det(h)}{\det(g)} = K(p)$

$\therefore K(p) = \frac{\langle d(fN)(v_1) \times d(fN)(v_2), N \rangle}{f^2}$