Assignment 3

1. let S be a surface in R^3 without boundary

let X, Y be two parameterization of S

then $\exists\ \mathcal{U}, \mathcal{U}'$, two open subsets of R^2 such that

$$X: \mathcal{U} \to S \qquad Y: \mathcal{U}' \to S$$ are both bijective

let $\delta = Y^{-1} \circ X : \mathcal{U} \to \mathcal{U}'$ (it is a bijective, and smooth)

$\therefore\ X = Y \circ \delta$

$$\therefore \begin{cases} \partial_u X = \partial_{u'} Y \cdot \dfrac{\partial u'}{\partial u} + \partial_{v'} Y \cdot \dfrac{\partial v'}{\partial u} \in \operatorname{span}\{\partial_{u'} Y, \partial_{v'} Y\} \\ \partial_v X = \partial_{u'} Y \cdot \dfrac{\partial u'}{\partial v} + \partial_{v'} Y \cdot \dfrac{\partial v'}{\partial v} \in \operatorname{span}\{\partial_{u'} Y, \partial_{v'} Y\} \end{cases}$$

$\therefore\ \operatorname{span}\{\partial_u X, \partial_v X\} \subseteq \operatorname{span}\{\partial_{u'} Y, \partial_{v'} Y\}$

$\therefore$ similarly, $\operatorname{span}\{\partial_{u'} Y, \partial_{v'} Y\} \subseteq \operatorname{span}\{\partial_u X, \partial_v X\}$

$\therefore$ the tangent spaces are the same.

2. 1°.

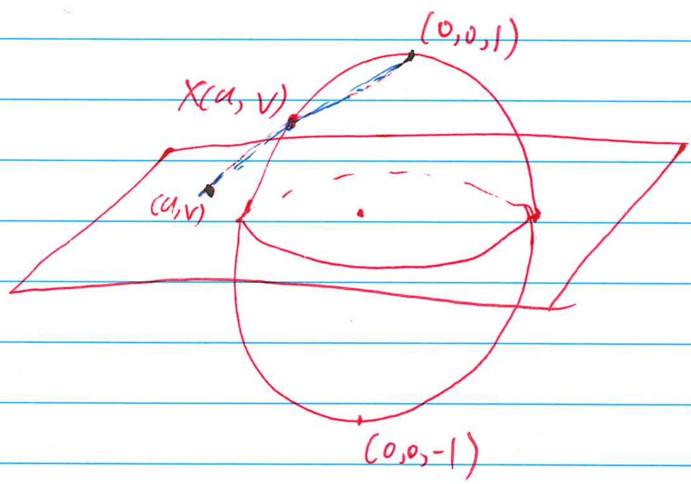
$$\frac{\partial X}{\partial u} = \left(\frac{2(1-u^2+v^2)}{(1+u^2+v^2)^2}, -\frac{4uv}{(1+u^2+v^2)^2}, \frac{4u}{(1+u^2+v^2)^2} \right)$$

$$\frac{\partial X}{\partial v} = \left(-\frac{4uv}{(1+u^2+v^2)^2}, \frac{2(1+u^2-v^2)}{(1+u^2+v^2)^2}, \frac{4v}{(1+u^2+v^2)^2} \right)$$

$$\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v} = \frac{1}{(1+u^2+v^2)^4} \left(-8u(1+u^2+v^2), -8v(1+u^2+v^2), 4(1-u^2-v^2)(1+u^2+v^2) \right)$$

$$\therefore\ \frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v} = 0 \iff \begin{cases} u = v = 0 \\ 1-u^2-v^2 = 0 \end{cases}$$ contradiction

$$\therefore\ \frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v} \neq 0$$



2°. check $X(u,v) \in x^2+y^2+z^2=1$

3°. if $X(u_1, v_1) = X(u_2, v_2)$

$$\frac{-1+u_1^2+v_1^2}{1+u_1^2+v_1^2} = \frac{-1+u_2^2+v_2^2}{1+u_2^2+v_2^2}$$

$$\left.\begin{array}{l} \Rightarrow u_1^2+v_1^2 = u_2^2+v_2^2 \\ \dfrac{2u_1}{1+u_1^2+v_1^2} = \dfrac{2u_2}{1+u_2^2+v_2^2} \end{array}\right\} \Rightarrow u_1 = u_2$$

similarly, $v_1 = v_2$

$\therefore\ X$ is injective.

4°、 check $(0,0,1) \notin X(R^2)$

5°、 check $X: R^2 \to S^2 \setminus (0,0,1)$ is surjective.

$\forall\ (x_0, y_0, z_0) \in S^2 \setminus (0,0,1)$

Solve $\left(\dfrac{2u}{1+u^2+v^2}, \dfrac{2v}{1+u^2+v^2}, \dfrac{-1+u^2+v^2}{1+u^2+v^2} \right) = (x_0, y_0, z_0)$

- $\dfrac{-1+u^2+v^2}{1+u^2+v^2} = z_0 \Rightarrow u^2+v^2 = \dfrac{z_0+1}{1-z_0}$
- $\dfrac{2u}{1+u^2+v^2} = x_0 \Rightarrow u = \dfrac{x_0}{1-z_0}$
- $\dfrac{2v}{1+u^2+v^2} = y_0 \Rightarrow v = \dfrac{y_0}{1-z_0}$

$\Big\}$ check $\left(\dfrac{x_0}{1-z_0}\right)^2 + \left(\dfrac{y_0}{1-z_0}\right)^2 + z_0^2 = 1$ using $x_0^2+y_0^2+z_0^2=1$

∴ such (u,v) exists

∴ X is onto to $S^2 \setminus (0,0,1)$

6°、

$$g_{ij} = \begin{bmatrix} \dfrac{4}{(1+u^2+v^2)^2} & 0 \\ 0 & \dfrac{4}{(1+u^2+v^2)^2} \end{bmatrix}$$

3、 $\alpha(t) = (\sin t \cos u_0, \sin t \sin u_0, \cos t) \quad t \in [a,b] \subseteq [0,\pi]$ with $\begin{cases} u(a) = a \\ u(b) = b \end{cases}$

$\beta(t) = X(u(t), v(t)), \quad t \in [a,b] \subseteq (0,\pi)$

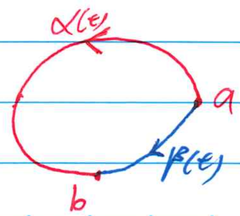
∴ $\beta'(t) = u'(t)\, \partial_u X + v'(t)\, \partial_v X, \quad \alpha'(t) = \partial_u (x)$

∴ $\displaystyle\int_a^b \langle \beta'(t), \partial_u X \rangle \leq \int_a^b |\beta'(t)| = L(\beta)$

$\displaystyle = \int_a^b u'(t) = u(b) - u(a) = b - a$ since u is smooth

$\displaystyle = \int_a^b |\partial_u X| = \int_a^b |\alpha'(t)| = L(\alpha)$

∴ $L(\alpha) \leq L(\beta)$



what if $a=0,\ b=\frac{3}{2}\pi$?

in this case, if we need $u(t)$ to be smooth

then $u(0)=0, \quad u(\frac{3}{2}\pi) = -\frac{\pi}{2} \neq \frac{3}{2}\pi$

∴ this argument doesn't work! and in this case, $L(\beta) < L(\alpha)$