Assignment 2.

1. 1°. Just check the coordinate of $X(u,v)$ satisfies $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

2°. $\because$ $\frac{\partial x}{\partial u} = (a\sinh u\cos v,\ b\sinh u\sin v,\ c\cosh u)$

$\frac{\partial x}{\partial v} = (-a\cosh u\sin v,\ b\cosh u\cos v,\ 0)$

$\because$ $c\cdot\cosh u \neq 0$ for any u

$\therefore$ $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}$ are linearly independent

$\therefore$ it is a regular surface.

3°. $(-\infty,\infty)\times(0,2\pi)$

2. 1°. $\alpha(s) = (\cos s, \sin s, 0)$

$w(s) = (-\sin s, \cos s, 1)$

$\therefore X(s,v) = (\cos s - v\sin s,\ \sin s + v\cos s,\ v)$

$\therefore (\cos s - v\sin s)^2 + (\sin s + v\cos s)^2 - v^2 = 1$

2°. it is surjective

for $s\in[0,2\pi)$, X is injective

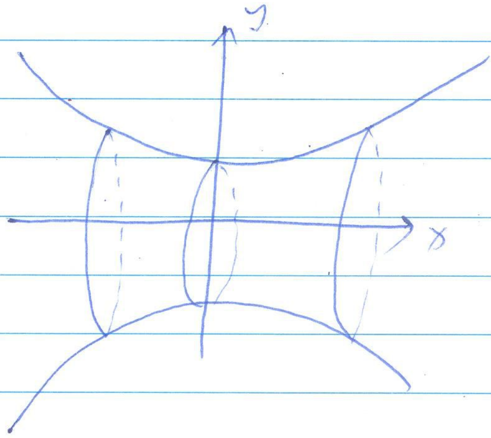
For fixed v, X maps S^1 to $x^2+y^2=1+v^2$ bijectively.

3°. $\frac{\partial x}{\partial s} = (-\sin s - v\cos s,\ \cos s - v\sin s,\ 0)$

$\frac{\partial x}{\partial v} = (-\sin s, \cos s, 1)$

$\therefore$ X has rank 2 since $\frac{\partial x}{\partial s}, \frac{\partial x}{\partial v}$ are linearly independent.

3. $y^2+z^2 = \cosh^2 x$



4. $X(u,v) = \left(u - \frac{u^3}{3} + uv^2,\ v - \frac{v^3}{3} + vu^2,\ u^2 - v^2\right)$

$\therefore \dfrac{\partial x}{\partial u} = (1 - u^2 + v^2,\ 2uv,\ 2u)$

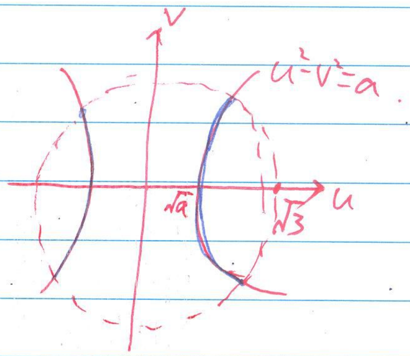
$\dfrac{\partial x}{\partial v} = (2uv,\ 1 - v^2 + u^2,\ -2v)$

1°. $\dfrac{\partial x}{\partial u} \times \dfrac{\partial x}{\partial v} = \left[-2u(1+u^2+v^2),\ 2v(1+u^2+v^2),\ 1-(u^2+v^2)^2\right]$

$\therefore \dfrac{\partial x}{\partial u} \times \dfrac{\partial x}{\partial v} = 0 \iff u = v = 0$ and $u^2+v^2=1$ contradiction

$\therefore X$ is regular

2°. When $u^2+v^2<3$



if $X(u_1, v_1) = X(u_2, v_2)$

then $u_1^2 - v_1^2 = u_2^2 - v_2^2 = a$ Assume $a \geq 0$

$\therefore (u_1, v_1), (u_2, v_2)$ lie on $u^2 - v^2 = a$

and $\begin{cases} u_1 - \frac{u_1^3}{3} + u_1 v_1^2 = u_2 - \frac{u_2^3}{3} + u_2 v_2^2 \\ v_1 - \frac{v_1^3}{3} + v_1 u_1^2 = v_2 - \frac{v_2^3}{3} + v_2 u_2^2 \end{cases}$

$\xrightarrow{u_i^2 - v_i^2 = a} \begin{cases} (1-a)u_1 + \frac{2}{3}u_1^3 = (1-a)u_2 + \frac{2}{3}u_2^3 & \text{— ①} \\ (1+a)v_1 + \frac{2}{3}v_1^3 = (1+a)v_2 + \frac{2}{3}v_2^3 & \text{— ②} \end{cases}$

For ②, let $g(v) = (1+a)v + \frac{2}{3}v^3 \Rightarrow g'(v) = 1 + a + 2v^2 > 0$ since $a \geq 0$

$\therefore g$ is increasing

$\therefore g(v_1) = g(v_2) \Rightarrow v_1 = v_2$

$\therefore u_i^2 - v_i^2 = a,\ v_1 = v_2$

$\therefore u_1 = \pm u_2$

if $u_1 \neq u_2$, then $u_1 = -u_2$

$\therefore$ ① $\Rightarrow (1-a)u_1 + \frac{2}{3}u_1^3 = -\left[(1-a)u_1 + \frac{2}{3}u_1^3\right]$

$\therefore u_1\left[(1-a) + \frac{2}{3}u_1^2\right] = 0$

$\therefore u_1 = 0 \Rightarrow u_1 = u_2 = 0$, contradiction

$\therefore u_1 \neq 0,\ 1 - a + \frac{2}{3}u_1^2 = 0$

$\therefore 0 = 1 - a + \frac{2}{3}u_1^2 \geq 1 - a + \frac{2}{3}a$ $[|u_1| \geq \sqrt{a}]$

$= 1 - \frac{a}{3} > 0$ since $a < 3$

$\therefore 0 > 0$, contradiction

$\therefore u_1 = u_2$ $\therefore X$ is injective for $u^2+v^2<3$ if $a \geq 0$

• For $a \le 0$, it is similar to $a \ge 0$

$\therefore$ X is injective on $u^2+v^2<3$.

3°. $X(\sqrt{3},0)=(0,0,3)$

$X(-\sqrt{3},0)=(0,0,3)$