Assignment 10.

(1) i) If M is diffeomorphic to a sphere
then $\chi(M) = 2 - 0$

$$\int_M K\,dA = 2\pi\chi(M) = 2\pi \cdot 2 = 4\pi$$

ii) If M is diffeomorphic to a torus
then $\chi(M) = 2 - 2\cdot 1 = 0$

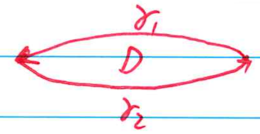
$$\therefore \int_M K\,dA = 2\pi \cdot \chi(M) = 0$$

(2) Suppose NOT.

Then we have two geodesic and they bound an open set which is diffeomorphic to a disk.

$\therefore$ By the Gauss-Bonnet Theorem.

$$\sum_{i=1}^{2} \int_{\gamma_i} k_g\,ds + \iint_D K\,dA + \sum_{j=1}^{2} \theta_j = 2\pi\chi(D)$$



$$\therefore 2\pi \cdot 1 = 0 + \iint_D K\,dA + \sum_{j=1}^{2} \theta_j$$

$$\leq 0 + \iint_D 0 \cdot dA + (\pi + \pi) = 2\pi \qquad (K \leq 0,\ -\pi \leq \theta_j \leq \pi)$$

$\therefore$ $K \equiv 0$ on D and $\theta_j = \pi$

$\Rightarrow$ $\gamma_1 = -\gamma_2$ which is a contradiction to γ_1 and γ_2 bound an open set.

$\therefore$ two geodesics from a point p cannot meet again so that they bound a simple region.

(3) (a) $\because$ $$\iint_M K\,dA = 2\pi\chi(M) = 4\pi(1-g)$$

$\therefore$ if M is not homeomorphic to a sphere then $g \geq 1$

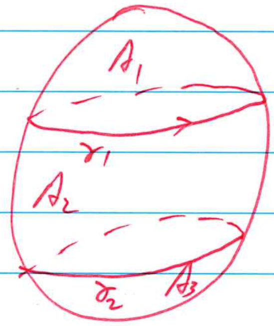
$\therefore$ $$0 \geq 4\pi(1-g) = \iint_M K\,dA > 0 \quad \text{since } K>0$$

which is a contradiction

$\therefore$ M is diffeomorphic to a sphere.

(b) Suppose we have two distinct simple closed geodesics on M and they do NOT intersect.

$\therefore$ γ_1, γ_2 divide M into three parts.



$\therefore$ $$\cancelto{0}{\int_{\gamma_1} k_g} + \int_{A_1} K\,dA = 2\pi\chi(A_1) = 2\pi$$

$$\cancelto{0}{\int_{\gamma_2} k_g} + \int_{A} K\,dA = 2\pi\chi(A_3) = 2\pi.$$

$\therefore$ $$\int_{A_1} K\,dA + \int_{A_3} K\,dA = 4\pi$$

$\because$ $$\int_M K\,dA = 4\pi$$

$\therefore$ $$\int_{A_3} K\,dA = 0$$

but $K>0$

$\therefore$ it is a contradiction

$\therefore$ γ_1 and γ_2 must intersect.