Solution of Midterm

(1) (a) From handout 1

(b) From handout 1

(c) [The proof is similar to the proof of Prop 2 in handout 2]

Suppose NOT.

then $\exists\ s_1^n, s_2^n, s_3^n, s_4^n \in (a,b)$ such that

$$\begin{cases} s_1^n, s_2^n, s_3^n, s_4^n \to s_0 \quad \text{and} \quad s_1^n \neq s_2^n \neq s_3^n \neq s_4^n \\ \alpha(s_1^n), \alpha(s_2^n), \alpha(s_3^n), \alpha(s_4^n) \text{ are on a plane.} \end{cases}$$

$\therefore\ \exists\ \vec{v}_n, \vec{n}_n$ such that $(|\vec{n}_n|=1)$

$$\langle \alpha(s_i^n) - \vec{v}_n, \vec{n}_n \rangle = 0, \quad \forall i=1,2,3,4, \quad \forall n \in \mathbb{N}.$$

let $f_n(t) = \langle \alpha(t) - \vec{v}_n, \vec{n}_n \rangle$

$\therefore$ by assumption, $f_n(s_1^n) = f_n(s_2^n) = f_n(s_3^n) = f_n(s_4^n) = 0$

$\therefore\ \exists\ a_1^n, a_2^n, a_3^n, b_1^n, b_2^n, C^n$ lie between $s_1^n, s_2^n, s_3^n, s_4^n$ such that

$$\begin{cases} f_n'(a_i^n) = 0 \\ f_n''(b_i^n) = 0 \\ f_n'''(C^n) = 0 \end{cases}$$

$$\therefore \begin{cases} \langle T(a_1^n), \vec{n}_n \rangle = \langle T(a_2^n), \vec{n}_n \rangle = \langle T(a_3^n), \vec{n}_n \rangle = 0 \\ \langle T'(b_1^n), \vec{n}_n \rangle = \langle T'(b_2^n), \vec{n}_n \rangle = 0 \\ \langle T''(C^n), \vec{n}_n \rangle = 0 \end{cases}$$

$\because\ T(a_1^n) \to T(s_0), \quad T'(b_1^n) \to k N(s_0)$

$\therefore\ \langle T(s_0), \vec{n}_n \rangle \to 0, \quad \langle k N(s_0), \vec{n}_n \rangle \to 0$

$\because\ k>0 \quad \therefore\ \langle \vec{n}_n, B(s_0) \rangle \to 1$ since $\{T(s_0), N(s_0), B(s_0)\}$ is an o.b.

$\because\ T''(C^n) \to T''(s_0) = k(-kT + \tau B) + k'N$ choose $\vec{n}_n$ to be the direction of $B(s_0)$

$\therefore\ \langle T''(C^n), \vec{n}_n \rangle = 0, \quad \langle \vec{n}_n, B(s_0) \rangle \to 1, \langle \vec{n}_n, T(s_0) \rangle \to 0, \langle \vec{n}_n, N(s_0) \rangle \to 0$

$\Rightarrow \langle k\tau B(s_0), B(s_0) \rangle \to 0$

$\Rightarrow \tau = 0$, contradiction to $\tau \neq 0$

$\therefore$ any four distinct points on α which are close enough to $\alpha(s_0)$ will not lie on a plane.

(2) Question 3 of Assignment 1

(3) (a) See the lecture notes.

(b) See the lecture notes.

(c) $\because E=G,\ F=0$

$\therefore H(p)=\frac{1}{2}\frac{eG-2fF+Eg}{EG-F^2}=\frac{1}{2}\cdot\frac{eG+Eg}{EG}=\frac{1}{2E}$

In the proof we use $\begin{cases} E(x)=G(x) & \forall x\in S \\ F(x)\equiv 0 \end{cases}$

$=\frac{1}{2E}(\langle N, X_{uu}\rangle+\langle N, X_{vv}\rangle)$ (how above we only assume $E(p)=G(p)$, $F(p)=0$ for fixed $p\in S$?)

$=\frac{\langle N, X_{uu}+X_{vv}\rangle}{2E}$

$\because E=G \Rightarrow \langle X_u, X_u\rangle=\langle X_v, X_v\rangle \Rightarrow \frac{d}{du}\langle X_u, X_u\rangle=\frac{d}{du}\langle X_v, X_v\rangle$

$\Rightarrow \langle X_{uu}, X_u\rangle=\langle X_{uv}, X_v\rangle$ —— ⊛

$F\equiv 0 \Rightarrow \frac{d}{du}\langle X_u, X_v\rangle=0 \Rightarrow \langle X_{uu}, X_v\rangle+\langle X_{uv}, X_u\rangle=0$

$\therefore \langle X_{uu}+X_{vv}, X_u\rangle=\langle X_{uu}, X_u\rangle+\langle X_{vv}, X_u\rangle=\langle X_{uu}, X_u\rangle+\underbrace{\frac{d}{dv}\langle X_u, X_v\rangle}_{=0,\ \because F\equiv 0}-\langle X_{uv}, X_v\rangle=0$ by ⊛

and similarly, $\langle X_{uu}+X_{vv}, X_v\rangle=0$

$\therefore X_{uu}+X_{vv}$ is parallel to N

$\therefore H(p)=0 \Leftrightarrow X_{uu}+X_{vv}=0$.

(4) Question 4 of Assignment 4.

(5) $X(\theta, w)=a(\theta)+vW(\theta)$

$\therefore X_\theta=a'(\theta)+vw'(\theta)$

$\because X_\theta=\lambda w$

$\therefore \langle a'(\theta)+vw'(\theta), w\rangle=\lambda\langle w,w\rangle=\lambda$

$=\langle(-\sin\theta, \cos\theta, 0), (\sin\tfrac{1}{2}\theta\cos\theta, \sin\tfrac{1}{2}\theta\sin\theta, \cos\tfrac{1}{2}\theta)\rangle$

$+v\langle w', w\rangle$

$=0+v\langle w', w\rangle$

$\therefore v\langle w', w\rangle=\lambda$

$\frac{v}{2}\langle w,w\rangle'=\lambda$

$\therefore \lambda=0$ since $|w|\equiv 1$

(6) (a) $X(u,v) = (\alpha(u)\cos v,\ \alpha(u)\sin v,\ \beta(u))$, $\quad (\alpha')^2+(\beta')^2=1,\ \alpha>0$

$\therefore\ X_u = \langle \alpha'\cos v,\ \alpha'\sin v,\ \beta' \rangle, \quad X_v = \langle -\alpha\sin v,\ \alpha\cos v,\ 0 \rangle$

$X_{uu} = \langle \alpha''\cos v,\ \alpha''\sin v,\ \beta'' \rangle, \quad N = (-\beta'\cos v,\ -\beta'\sin v,\ \alpha')$

$X_{uv} = \langle -\alpha'\sin v,\ \alpha'\cos v,\ 0)$

$X_{vv} = \langle -\alpha\cos v,\ -\alpha\sin v,\ 0)$

$$\therefore\ g = \begin{bmatrix} (\alpha')^2+(\beta')^2 & 0 \\ 0 & \alpha^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \alpha^2 \end{bmatrix}$$

$$h = \begin{bmatrix} -\alpha''\beta'+\alpha'\beta'' & 0 \\ 0 & \alpha\beta' \end{bmatrix}$$

$$\therefore\ K = \frac{\det(h)}{\det(g)} = \frac{\alpha\beta'(-\alpha''\beta'+\alpha'\beta'')}{\alpha^2}$$

$$= \frac{\alpha'\beta'\beta''-\alpha''(\beta')^2}{\alpha} = \frac{\alpha'\beta'\beta''-\alpha''(1-(\alpha')^2)}{\alpha}$$

$$= \frac{-\alpha''+\alpha'(\alpha'\alpha''+\beta'\beta'')}{\alpha} = -\frac{\alpha''}{\alpha}+\frac{\alpha'}{\alpha}\cdot\frac{1}{2}((\alpha')^2+(\beta')^2)'$$

$$= -\frac{\alpha''}{\alpha} \quad \text{since } (\alpha')^2+(\beta')^2\equiv 1$$

(b) if $K\equiv 0$

then $\alpha''\equiv 0$

$\therefore\ \alpha(u) = au+b$ for some a, b

$\alpha'(u) = a$

$\therefore\ (\beta')^2 = \sqrt{1-a^2}$

$\therefore\ \beta(u) = cu+d$ for some c, d

$\therefore\ (\alpha(u), \beta(u))$ is a straight line in $\mathbb{R}^2$.