

Möbius strip

$$\mathbf{X}(\theta, v) = (\cos \theta, \sin \theta, 0) + v(\sin \frac{1}{2}\theta \cos \theta, \sin \frac{1}{2}\theta \sin \theta, \cos \frac{1}{2}\theta) = \mathbf{a}(\theta) + v\mathbf{w}(\theta)$$

$-\pi < \theta < \pi$, $-\frac{1}{2} < v < \frac{1}{2}$. When $\theta = \pi$, $\mathbf{w}(\theta) = (-1, 0, 0)$. When $\theta = -\pi$, then $\mathbf{w}(\theta) = (1, 0, 0)$.

Tangent space and the first fundamental form

1. TANGENT SPACE

Definition 1. Let $\mathbf{X} : U \rightarrow \mathbb{R}^3$ be a regular surface patch, and let $M = \mathbf{X}(U)$. Let $p \in M$ be a point in the surface. That is $p = \mathbf{X}(u_0^1, u_0^2)$ for some (u_0^1, u_0^2) in U . Then the *tangent space* $T_p(M)$ of M at p is the vector space spanned by $\mathbf{X}_1(u_0^1, u_0^2), \mathbf{X}_2(u_0^1, u_0^2)$. Since $\mathbf{X}_1, \mathbf{X}_2$ are linearly independent, $\dim(T_p(M)) = 2$.

Here $\mathbf{X}_1 = \frac{\partial \mathbf{X}}{\partial u^1}$, etc.

Proposition 1. $T_p(M)$ is well defined. Namely, suppose $\phi : V \rightarrow U$ is a diffeomorphism, $V \subset \mathbb{R}^2$ with coordinates (v^1, v^2) . Let $\mathbf{Y} = \mathbf{X} \circ \phi$. Then the vector space spanned by $\frac{\partial \mathbf{X}}{\partial u^1}, \frac{\partial \mathbf{X}}{\partial u^2}$, and the vector space spanned by $\frac{\partial \mathbf{Y}}{\partial v^1}, \frac{\partial \mathbf{Y}}{\partial v^2}$ are the same.

Proposition 2. Let $\mathbf{X} : U \rightarrow \mathbb{R}^3$ be a regular surface patch and let $M = \mathbf{X}(U)$. Let $\alpha(t)$ be a smooth curve in $\mathbb{R}^3$ such that $\alpha(t) \in M$ for all $t \in (a, b)$ passing through a point $p = \alpha(t_0)$ say. Then there is $\epsilon > 0$ and there is a unique smooth curve $\beta(t)$ in U with $t \in (t_0 - \epsilon, t_0 + \epsilon)$ such that $\alpha(t) = \mathbf{X}(\beta(t))$ in $(t_0 - \epsilon, t_0 + \epsilon)$.

To prove the proposition, we need the following fact:

Lemma 1. Let $\mathbf{X} : U \rightarrow \mathbb{R}^3$ be a regular surface patch. For any $\mathbf{u}_0 = (u_0^1, u_0^2) \in U$, there is an open set $V \subset U$, such that the regular surface patch $\mathbf{X} : V \rightarrow \mathbb{R}^3$ has a reparametrization so that the surface can be realized as the graph of a function over a coordinate plane.

Sketch of proof. Let $\mathbf{X}(u^1, u^2) = (x(u^1, u^2), y(u^1, u^2), z(u^1, u^2))$. Since the surface patch is regular we may assume that $\det \begin{pmatrix} x_{u^1} & x_{u^2} \\ y_{u^1} & y_{u^2} \end{pmatrix} \neq 0$.

Define the map $F : U \rightarrow \mathbb{R}^2$ with $F(u^1, u^2) = (x(u^1, u^2), y(u^1, u^2))$. Then $dF_{\mathbf{u}_0}$ is nonsingular. Hence there exist open sets V, W with $\mathbf{u}_0 \in V \subset U$, $(x_0, y_0) = F(u_0^1, u_0^2) \in W$. Hence $\mathbf{Y} = \mathbf{X} \circ F^{-1} :$

$W \rightarrow \mathbb{R}^3$ is a reparametrization of $\mathbf{X} : V \rightarrow \mathbb{R}^3$. If $F^{-1}(x_1, y_1) = (u^1(x_1, y_1), u^2(x_1, y_1))$, then

$$\begin{aligned} \mathbf{Y}(x_1, y_1) &= (x(u^1(x_1, y_1), u^2(x_1, y_1)), y(u^1(x_1, y_1), u^2(x_1, y_1)), z(u^1(x_1, y_1), u^2(x_1, y_1))) \\ &= (x_1, y_1, f(x_1, y_1)) \end{aligned}$$

with $f(x_1, y_1) = z(u^1(x_1, y_1), u^2(x_1, y_1))$. □

Sketch of proof of Proposition 2. Let α, p as in the proposition and let $(u_0^1, u_0^2) \in U$ with $\mathbf{X}(u_0^1, u_0^2) = p$. By the lemma, we may assume that near p , the surface is a graph over xy -plane. Namely, there are open sets $\mathbf{u}_0 \in V \subset U$ and W and a diffeomorphism $\phi : W \rightarrow V$ with $\phi^{-1}(\mathbf{u}_0) = (x_0, y_0) \in W$ such that $\mathbf{Y}(x, y) = \mathbf{X} \circ \phi(x, y) = (x, y, f(x, y))$. Now $\alpha(t) \in \mathbf{X}(U)$ so $\alpha(t) = (x(t), y(t), f(x(t), y(t))) = \mathbf{Y}(x(t), y(t))$. Let $\beta(t) = \phi(x(t), y(t))$. Then $\mathbf{X}(\beta(t)) = \alpha(t)$. □

Corollary 1. Let $\mathbf{X} : U \rightarrow \mathbb{R}^3$ be a regular surface patch, and let $M = \mathbf{X}(U)$. Let $p \in M$ be a point in the surface. Then $T_p(M)$ consists of the tangent vectors of smooth curves on M passing through p .

Definition 2. Let $\mathbf{X} : U \rightarrow \mathbb{R}^3$ be a regular surface patch and let $M = \mathbf{X}(U)$. A nonzero vector N at a point $p = \mathbf{X}(u^1, u^2) \in M$ is called a *normal vector* of M at p if it is orthogonal to $T_p(M)$. A normal vector N at p is called a *unit normal vector* if N has unit length.

Questions: How many normal vectors at a point are there? How many unit normal vectors?

2. FIRST FUNDAMENTAL FORM

Definition 3. Let $\mathbf{X} : U \rightarrow \mathbb{R}^3$ be a regular surface patch, and let $M = \mathbf{X}(U)$. Let $p \in M$ be a point in the surface. The *first fundamental form* g of M at p is the inner product at each $T_p(M)$ given by $g(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle$. The first fundamental form of M is the inner product given by $g(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w} \rangle$ on every $T_p(M)$ for with $p \in M$.

Let $\mathbf{X} : U \rightarrow V \subset M$ be a coordinate parametrization. The *coefficients of the first fundamental form* g with respect to the parametrization are defined as:

$$\begin{cases} E = g(\mathbf{X}_u, \mathbf{X}_u) = \langle \mathbf{X}_u, \mathbf{X}_u \rangle; \\ F = g(\mathbf{X}_u, \mathbf{X}_v) = \langle \mathbf{X}_u, \mathbf{X}_v \rangle; \\ G = g(\mathbf{X}_v, \mathbf{X}_v) = \langle \mathbf{X}_v, \mathbf{X}_v \rangle. \end{cases}$$

if (u, v) denotes points in U .

If we use (u^1, u^2) instead of (u, v) and let $\mathbf{X}_i = \frac{\partial \mathbf{X}}{\partial u^i}$, then we also denote coefficients of the first fundamental form g as

$$g_{ij} = \langle \mathbf{X}_i, \mathbf{X}_j \rangle.$$

Suppose $\alpha(t) = (x(t), y(t), z(t))$ is a smooth curve on M , $a \leq t \leq b$ such that $\alpha(t) = \mathbf{X}((u(t), v(t)))$ in local coordinates. Then the length of α is given by

$$\begin{aligned} \ell &= \int_a^b |\alpha'(t)| dt \\ &= \int_a^b \left(E(\alpha(t)) \left(\frac{du}{dt} \right)^2 + 2F(\alpha(t)) \frac{du}{dt} \frac{dv}{dt} + G(\alpha(t)) \left(\frac{dv}{dt} \right)^2 \right)^{\frac{1}{2}} dt \\ &= \int_a^b \left((E(u')^2 + 2Fu'v' + G(v')^2) \right)^{\frac{1}{2}} dt. \end{aligned}$$

If we use (u^1, u^2) instead of (u, v) and $\mathbf{X}_i = \frac{\partial \mathbf{X}}{\partial u^i}$,

$$\ell = \int_a^b \left(\sum_{i,j=1}^2 g_{ij} (u^i)' (u^j)' \right)^{\frac{1}{2}} dt.$$

So sometimes, the first fundamental form is written symbolically as $ds^2 = Edu^2 + 2Fdu dv + Gdv^2$, or $g = \sum_{i,j=1}^2 g_{ij} du^i du^j$.

Assignment 3, Due 27/9/2018

- (1) Prove that the definition of tangent space is independent of the choice of parametrization.
- (2) Consider the stereographic projection by $\mathbf{X} : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by

$$\mathbf{X}(u, v) = \left(\frac{2u}{1 + u^2 + v^2}, \frac{2v}{1 + u^2 + v^2}, \frac{-1 + u^2 + v^2}{1 + u^2 + v^2} \right).$$

Show that this is a regular surface patch with image being the unit sphere $\mathbb{S}^2$ with the north pole $(0, 0, 1)$ deleted. Find also the coefficients of the first fundamental form.

- (3) Consider the sphere parametrized by spherical coordinates:

$$\mathbf{X}(u, v) = (\sin v \cos u, \sin v \sin u, \cos v)$$

with $-\pi < u < \pi, 0 < v < \pi$. Find the length of the curve α given by $u = u_0$ and $a \leq v \leq b$ with $0 < a < b < \pi$. (That is $\alpha(t) = (\sin t \cos u_0, \sin t \sin u_0, \cos t)$, with $a \leq t \leq b$.) Let $\beta(t)$ be another curve joining $\alpha(a)$ to $\alpha(b)$ on the surface, i.e. $\beta(t) = \mathbf{X}(u(t), v(t))$, $a \leq t \leq b$ with $\beta(a) = \alpha(a)$, $\beta(b) = \alpha(b)$. Show that $\ell(\beta) \geq \ell(\alpha)$.