Tutorial 9.

1. Prove that a curve $C \subseteq S$ is both an asymptotic curve and a geodesic if and only if C is a (segment of a) straight line.

Pf: Let $\alpha(t)$ be a regular curve on S p.b.a.l.

$\because$ $\alpha''(t) = k_g \vec{n} + k_n \vec{N}$

$\therefore$ $\alpha(t)$ is asymptotic and geodesic

$\Leftrightarrow$ $k_n = 0$ and $k_g = 0$

$\Leftrightarrow$ $\alpha''(t) = 0$

$\Leftrightarrow$ α is a (segment of a) straight line.

2. Show that the straight lines are the only geodesics of a plane.

Pf: Let $\alpha(t)$ be a geodesic in a plane.

$\therefore$ $\alpha''(t) = k_g \vec{n} + k_n \vec{N}$, $k_g = 0$, $\vec{N}$ is constant

$\because$ $k_n = \langle \alpha''(t), \vec{N} \rangle$

$= \langle \alpha'(t), S_p(\alpha'(t)) \rangle$

$= \langle \alpha'(t), 0 \rangle$ since $\vec{N}$ is constant

$= 0$

$\therefore$ $\alpha''(t) \equiv 0$

$\therefore$ $\alpha'(t)$ is a constant vector

$\therefore$ $\alpha(t)$ is a straight line.

3、 Show that if all the geodesics of a connected open surface are plane curves, then the surface is contained in a plane or a sphere.

pf: [We prove that each point of the surface umbilical.]

$\forall\ P \in S$, $\forall\ \vec{v} \in T_pS$ with $|\vec{v}|=1$

$\exists$ a geodesic $\alpha_{\vec{v}}: (-\varepsilon, \varepsilon) \to S$ such that

$\alpha_{\vec{v}}(0) = P$, $\alpha_{\vec{v}}'(0) = \vec{v}$

$\therefore\ \alpha''_{\vec{v}}(0) = k_g \vec{n} + k_n \vec{N} = k_n \vec{N} = \langle \vec{v}, S_p(\vec{v}) \rangle \vec{N}$

$= (\cos^2\theta\, k_1 + \sin^2\theta\, k_2) \vec{N}$

where $k_1, k_2, \vec{v}_1, \vec{v}_2$ are the corresponding eigenvalus and eigenvectors. and $\vec{v} = \cos\theta\, \vec{v}_1 + \sin\theta\, \vec{v}_2$

Case 1. $\exists$ five different $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \in [0, 2\pi)$ such that

$\alpha''_{\vec{v}_i} = 0$ where $\vec{v}_i = \cos\theta_i\, \vec{v}_1 + \sin\theta_i\, \vec{v}_2$

$\therefore\ \cos^2\theta_i\, k_1 + \sin^2\theta_i\, k_2 = 0$ for $i = 1, 2, 3, 4, 5$

But if $k_1 \neq 0$, there are at most four $\theta \in [0, 2\pi)$ such that $\cos^2\theta\, k_1 + \sin^2\theta\, k_2 = 0$

$\therefore\ k_1 = 0$

similarly, $k_2 = 0$

$\therefore$ P is umbilical

Case 2: If there are at most four $\theta \in [0, 2\pi)$ with $\cos^2\theta\, \vec{v}_1 + \sin^2\theta\, \vec{v}_2 = 0$

then for other $\vec{v} \in T_pS$, $\alpha''_{\vec{v}}(0) \neq 0$

$\because\ \alpha''_{\vec{v}} = k_n \vec{N}$, $k_n \neq 0$

$\therefore$ By choosing the direction of $\vec{N}$ such that $k_n > 0$

$\therefore\ k_n(0) = k(0)$ where k is the curvature of α as a space curve

and $\vec{N} = \vec{N}_\alpha$ where $\vec{N}_\alpha$ is the normal vector of α, $\vec{N}_\alpha = \frac{\alpha''}{|\alpha''|}$

$\therefore\ S_p(\vec{v}) = S_p(\alpha'(0)) = \frac{d}{dt}(\vec{N} \circ \alpha)\Big|_{t=0} = \frac{d}{dt}(\vec{N}_\alpha \circ \alpha)\Big|_{t=0}$

$= \vec{N}_\alpha'(0) = -k\alpha'(0) + \tau(\alpha'(0) \times \vec{N}_\alpha(0))$

$= -k\alpha'(0)$ since α is a plane curve $\Rightarrow \tau = 0$

$\therefore\ \vec{v}$ is a principle direction

$\therefore$ all most all $\vec{v} \in T_pS$, $\vec{v}$ is a principle direction

$\therefore$ P is umbilical.

$\therefore$ By Prop 4 of handout.6, S is contained in a plane or a sphere.