Tutorial 8.

1. Does there exist a parametrization $X(u,v): U \to \mathbb{R}^3$ of a surface S such that $(g_{ij}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $(h_{ij}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$?

Ans: No.

Suppose there exists such S.

then, • $K = \dfrac{\det(h)}{\det(g)} = \dfrac{-1}{1} = -1$

• By Theorema Egregium of Gauss

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right]$$

$$= 0$$

∴ it is a contradiction.

∴ Such surface doesn't exist.

• Remark: this means there are some restriction on (g_{ij}) and (h_{ij}).
This is so called Gauss - Codazzi eq^n.

$$\begin{cases} \partial_k \Gamma^l_{ij} - \partial_j \Gamma^l_{ik} + \Gamma^p_{ij}\Gamma^l_{pk} - \Gamma^p_{ik}\Gamma^l_{pj} = g^{lq}(h_{ij}h_{kq} - h_{ik}h_{jq}) \\ \partial_k h_{ij} - \partial_j h_{ik} + \Gamma^p_{ij} h_{pk} - \Gamma^p_{ik} h_{pj} = 0 \end{cases}$$

for $\forall\, i, j, k, q$.

2. Consider the Poincaré disk model (D, g_{ij}) which is an abstract Riemannian surface with $D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1\}$ and

$$(g_{ij}) = \frac{4}{[1-(x^2+y^2)]^2}\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

(a) Compute all the Christoffel symbols Γ_{ij}^k and the Gauss curvature K

(b) Write down the geodesic eq^n and describe all the geodesics in (D, g_{ij})

pf: (a) $\Gamma_{ij}^k = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$

- $$\Gamma_{11}^k = \frac{1}{2}\cdot\frac{[1-(x^2+y^2)]^2}{4}\cdot\begin{bmatrix}1&0\\0&1\end{bmatrix}\cdot\frac{16}{[1-(x^2+y^2)]^3}\left(\begin{bmatrix}x\\0\end{bmatrix}+\begin{bmatrix}x\\0\end{bmatrix}-\begin{bmatrix}x\\y\end{bmatrix}\right)$$
$$= \frac{2}{1-(x^2+y^2)}\begin{bmatrix}x\\-y\end{bmatrix}$$

- $$\Gamma_{12}^k = \frac{1}{2}\cdot\frac{[1-(x^2+y^2)]^2}{4}\cdot\begin{bmatrix}1&0\\0&1\end{bmatrix}\cdot\frac{16}{[1-(x^2+y^2)]^3}\left(\begin{bmatrix}0\\x\end{bmatrix}+\begin{bmatrix}y\\0\end{bmatrix}-\begin{bmatrix}0\\0\end{bmatrix}\right)$$
$$= \frac{2}{1-(x^2+y^2)}\begin{bmatrix}y\\x\end{bmatrix}$$

- $$\Gamma_{22}^k = \frac{1}{2}\cdot\frac{[1-(x^2+y^2)]^2}{4}\cdot\begin{bmatrix}1&0\\0&1\end{bmatrix}\cdot\frac{16}{[1-(x^2+y^2)]^3}\left(\begin{bmatrix}0\\y\end{bmatrix}+\begin{bmatrix}0\\y\end{bmatrix}-\begin{bmatrix}x\\y\end{bmatrix}\right)$$
$$= \frac{2}{1-(x^2+y^2)}\begin{bmatrix}-x\\y\end{bmatrix}$$

- $\because E=G,\ F=0$

$\therefore$ let $f = \frac{1}{2}\lg E = \frac{1}{2}\lg\frac{4}{[1-(x^2+y^2)]^2} = -\lg\frac{1-(x^2+y^2)}{2}$

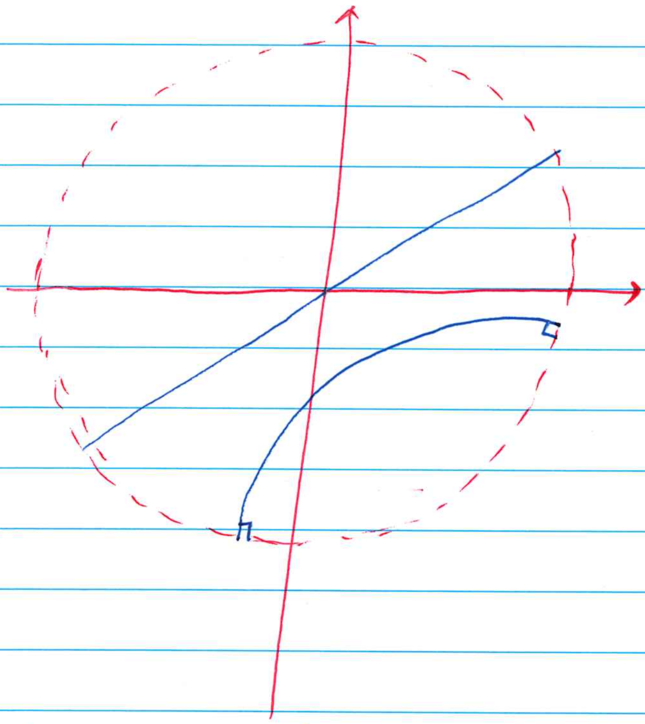
$$\therefore f_x = \frac{2x}{1-(x^2+y^2)},\quad f_{xx} = \frac{2[1-(x^2+y^2)]+4x^2}{[1-(x^2+y^2)]^2} = \frac{2(1+x^2-y^2)}{[1-(x^2+y^2)]^2}$$

$$f_y = \frac{2y}{1-(x^2+y^2)},\quad f_{yy} = \frac{2(1-x^2+y^2)}{[1-(x^2+y^2)]^2}$$

$$\therefore K = -e^{-2f}\Delta f = -\frac{[1-(x^2+y^2)]^2}{4}\cdot\frac{4}{[1-(x^2+y^2)]^2} = -1$$

(b) Geodesic eq^n: $\begin{cases} \ddot{u}^1 + \Gamma^1_{ij} \dot{u}^i \dot{u}^j = 0 \\ \ddot{u}^2 + \Gamma^2_{ij} \dot{u}^i \dot{u}^j = 0 \end{cases}$

$$\therefore \begin{cases} x''(t) + \dfrac{2x}{1-(x^2+y^2)}(x')^2 + \dfrac{4y}{1-(x^2+y^2)} x'y' + \dfrac{-2x}{[1-(x^2+y^2)]}(y')^2 = 0 \\ y''(t) + \dfrac{2y}{1-(x^2+y^2)}(y')^2 + \dfrac{4x}{1-(x^2+y^2)} x'y' + \dfrac{-2y}{1-(x^2+y^2)}(x')^2 = 0 \end{cases}$$



two types of geodesic:

1°, diameters

2°, part of a circle which is orthogonal to ∂D.