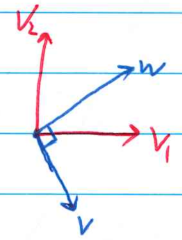
Tutorial 7.

1. Show that the sum of the normal curvatures for any pair of orthogonal directions, at a point $p \in S$, is a constant.



pf: let V_1, V_2, k_1, k_2 be the corresponding eigenvectors and eigenvalues of S_p

and $\{V_1, V_2\}$ is an orthonormal basis of T_pS.

$\therefore \forall\ V, W \in T_pS$ with $V \perp W$, $|V| = |W| = 1$

$\exists\ \theta \in [0, 2\pi)$ such that

$V = \cos\theta\, V_1 + \sin\theta\, V_2$, $\quad W = \cos(\theta + \frac{\pi}{2}) V_1 + \sin(\theta + \frac{\pi}{2}) V_2$

$\qquad\qquad\qquad\qquad\qquad = -\sin\theta\, V_1 + \cos\theta\, V_2$

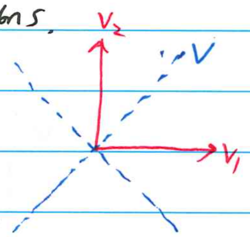
$\therefore k_n(V) = \langle V, S_p(V) \rangle$

$\qquad = \cos^2\theta\, k_1 + \sin^2\theta\, k_2$

$k_n(W) = \sin^2\theta\, k_1 + \cos^2\theta\, k_2$

$\therefore \dfrac{k_n(V) + k_n(W)}{2} = \dfrac{k_1 + k_2}{2} = H(p)$

2. Show that if the mean curvature is zero at a nonplannar point, then this point has two orthogonal asymptotic directions.



pf: let V_1, V_2, k_1, k_2 be the corresponding eigenvectors and eigenvalues of S_p.

and $\{V_1, V_2\}$ is an orthonormal basis of T_pS.

$\because H(p) = 0 \Rightarrow k_1 + k_2 = 0 \Rightarrow k_1 = -k_2$ $\Big\}$

$\qquad p$ is a nonplannar point $\Big\} \Rightarrow k_1 \neq 0, k_2 \neq 0$

let $V \in T_pS$ be a asymptotic direction and $|V| = 1$

$\therefore 0 = k_n(V) = \langle V, S_p(V) \rangle$

$\qquad = \langle \cos\theta\, V_1 + \sin\theta\, V_2,\ S_p(\cos\theta\, V_1 + \sin\theta\, V_2) \rangle$

$\qquad = \cos^2\theta\, k_1 + \sin^2\theta\, k_2$

$\qquad = \cos^2\theta\, k_1 - \sin^2\theta\, k_1$

$\because k_1 \neq 0 \quad \therefore \cos^2\theta = \sin^2\theta$

$\qquad \therefore \theta = \frac{\pi}{4}$ or $\frac{3}{4}\pi$ or $\frac{5}{4}\pi$ or $\frac{7}{4}\pi$

3. [Local convexity and curvature]

A surface $S \subseteq \mathbb{R}^3$ is locally convex at a point $P \in S$ if there exists a neighborhood $V \subseteq S$ of p such that V is contained in one of the closed half-spaces determined by $T_p(S)$ in $\mathbb{R}^3$.

If in addition, V has only one common point with $T_p(S)$, then S is called strictly locally convex at p.

(a) prove that S is strictly locally convex at P if the principle curvatures of S at p are nonzero with the same sign. (i.e. $K(p)>0$)

(b) Prove that if S is locally convex at p, then the principle curvatures at P do not have different signs. (thus, $K(p) \geq 0$)

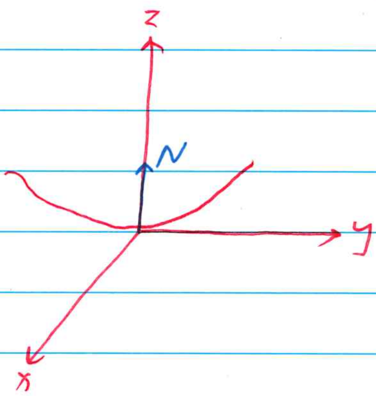
(c) To show that $K \geq 0$ does not imply local convexity, consider $(x,y) = x^3(1+y^2)$ defined on $\mathcal{U} = \{(x,y) : y^2 < \frac{1}{2}\}$

Show $K(p) \geq 0$ for $\forall p \in \mathcal{U}$

but the surface is not locally convex at $(0,0)$.

(d) The example of part (c) is also very special in the following local sense. Let P be a point in a surface S, and assume that there exists a neighborhood $V \subseteq S$ of p such that the principle curvatures on V do not have different signs

Show that S is locally convex at P.



pf: After rotation and translation

We can assume $P = (0,0,0)$

and S is a graph $(x, y, G(x,y))$ near P.

and $\partial_x S(0,0) = (1,0,0)$

$\partial_y S(0,0) = (0,1,0)$

$\vec{N}(0,0) = (0,0,1)$, $\vec{N} = (f,g,h)$ near P.

and $\partial_x S(0,0)$, $\partial_y S(0,0)$ are the eigenvectors of S_p corresponding to k_1, k_2.

(a) $\because K(P)>0 \quad \therefore k_1 \cdot k_2 > 0$

$k_1 = \langle \partial_x S(0,0), -\partial_x \vec{N}(0,0) \rangle = \langle \partial_x^2 S(0,0), \vec{N}(0,0) \rangle = \langle (0,0,G_{xx}(0,0)), (0,0,1) \rangle$

$= G_{xx}(0,0)$

similarly, $k_2 = G_{yy}(0,0)$

$\therefore G_{xx}(0,0) \cdot G_{yy}(0,0) > 0$

$\because$ $G_x(0,0) = G_y(0,0) = 0$

$\therefore$ $\begin{cases} \text{if } G_{xx}(0,0) > 0 \Rightarrow G_{yy}(0,0) > 0 \Rightarrow G(x,y) \geq 0 \text{ near } (0,0) \\ \qquad \text{and } G(x,y) > 0 \text{ for } (x,y) \neq (0,0) \\ \text{if } G_{xx}(0,0) < 0 \Rightarrow G_{yy}(0,0) < 0 \\ \qquad \Rightarrow G(x,y) \leq 0 \text{ near } (0,0) \\ \qquad \text{and } G(x,y) < 0 \text{ for } (x,y) \neq (0,0) \end{cases}$

$\therefore$ In any case, S is strictly locally convex at p.

(b) Since S is locally convex at p

$\therefore$ we can assume $G \geq 0$ near $(0,0)$

$\therefore$ $\begin{cases} G_{xx}(0,0) \geq 0 \\ G_{yy}(0,0) \geq 0 \end{cases}$ using $f''(x) = \lim\limits_{t \to 0} \dfrac{f(x+t) + f(x-t) - 2f(x)}{t^2}$

$\because$ $\langle \vec{N}, \partial_x S \rangle \equiv 0$

$\therefore$ $f + hG_x \equiv 0$

$\therefore$ $f_x + h_x G_x + hG_{xx} \equiv 0$

$$\begin{aligned} \therefore k_1 &= \langle \partial_x S, -\partial_x N \rangle |_{(0,0)} \\ &= -(f_x + h_x G_x)|_{(0,0)} \\ &= hG_{xx}(0,0) \\ &= 1 \cdot G_{xx}(0,0) \\ &\geq 0 \end{aligned}$$

Similarly, $k_2 \geq 0$

$\therefore$ $K(p) = k_1 \cdot k_2 \geq 0$

(c) S is $(x, y, G(x,y))$

then $\partial_x S = (1, 0, G_x)$ $\qquad \partial_{xx} S = (0, 0, G_{xx})$ $\qquad \partial_{xy} S = (0, 0, G_{xy})$

$\partial_y S = (0, 1, G_y)$ $\qquad \partial_{yy} S = (0, 0, G_{yy})$

$$\vec{N} = \frac{(-G_x, -G_y, 1)}{\sqrt{G_x^2 + G_y^2 + 1}}$$

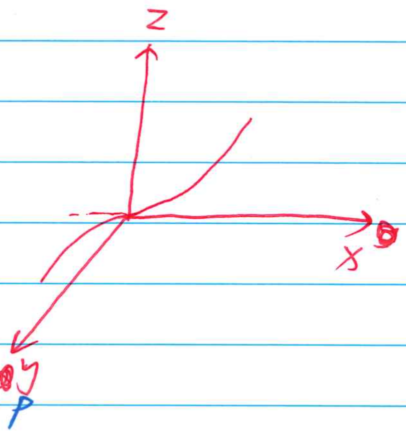
$$\therefore g = \begin{bmatrix} 1 + G_x^2 & G_x G_y \\ G_x G_y & 1 + G_y^2 \end{bmatrix} \qquad h = \begin{bmatrix} G_{xx} & G_{xy} \\ G_{xy} & G_{yy} \end{bmatrix} \cdot \frac{1}{\sqrt{1 + G_x^2 + G_y^2}}$$

$\therefore K = \dfrac{\det(h)}{\det(g)} = \dfrac{G_{xx}G_{yy} - G_{xy}^2}{(1+G_x^2+G_y^2)^2}$

$\because G(x,y) = x^3(1+y^2)$

$\therefore G_{xx} = 6x(1+y^2), \quad G_{xy} = 6x^2y, \quad G_{yy} = 2x^3$

$$\therefore G_{xx}G_{yy} - G_{xy}^2 = 12x^4(1+y^2) - 36x^4y^2$$
$$= 12x^4 - 24x^4y^2$$
$$= 12x^4(1-2y^2)$$



$\therefore K \geq 0$ on $U = \{(x,y) : y^2 < \frac{1}{2}\}$

- For $(x,y) = (0,0)$

  $G_x = (1,0,0), \quad G_y = (0,1,0)$

$\therefore \{(x,y,z) : z=0\}$ is the tangent plane of S at p

But S is not locally convex at p.

(d) The assumption means $k_1(x), k_2(x)$ have the same sign on V

it is stronger than $K(x) \geq 0$ for $\forall x \in V$.

(for $K(x) \geq 0$, we may have $k_1(x_1) > 0, \ k_2(x_1) > 0$
$k_1(x_2) < 0, \ k_2(x_2) < 0$)

W.L.O.G, we can assume $k_1(x) \geq 0, k_2(x) \geq 0$ for $\forall x \in V$

$\therefore \langle \vec{v}, -\partial_{\vec{v}} \vec{N}(x) \rangle \geq 0$ for $\forall \vec{v} \in TS$, $\forall x \in V$

$\therefore$ Any (x,y) near $(0,0)$

We can rotate the graph of $G(x,y)$ along z-axis

such that (x,y) becomes $(r,0)$ where $r = \sqrt{x^2+y^2}$

the new graph is $\tilde{G}(x,y)$

we show that $\tilde{G}(r,0) \geq 0$

$\because \{(x,y,0)\}$ is tangent to the graph of $\tilde{G}$ at $(0,0)$

$\therefore \tilde{G}_x(0,0) = 0$

$\because \langle \partial_x \xi, -\partial_x \tilde{N} \rangle \geq 0$

$= \langle \partial_x^2 \xi, \vec{N} \rangle$

$= \tilde{G}_{xx} \cdot \dfrac{1}{\sqrt{1+\tilde{G}_x^2+\tilde{G}_y^2}}$

$\therefore \tilde{G}_{xx} \geq 0$ near $(0,0)$

$\because \tilde{G}_x(0,0) = 0$. $\quad \therefore \tilde{G}_x(t,0) \geq 0$ for $t \geq 0$.

$\tilde{G}_x(t,0) \leq 0$ for $t \leq 0$

$\because \tilde{G}(0,0)=0$

$\therefore \tilde{G}(x,0)\geq 0$ for x closed to 0.

$\therefore \tilde{G}(r,0)\geq 0$ for r closed to 0

$\therefore G(x,y)\geq 0$ for $\forall (x,y)$ near 0

$\therefore S$ is locally convex at p.