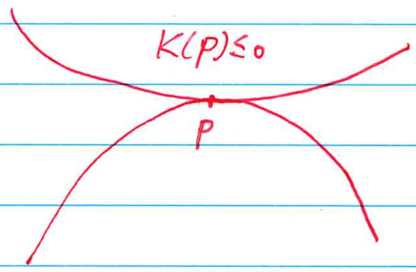
Tutorial 6.

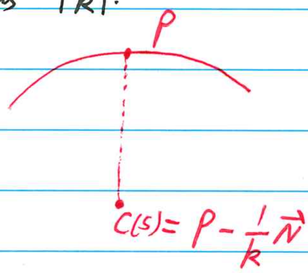
1. Let S be a regular surface without boundary.
Suppose $K(p) \le 0$ for $\forall p \in S$
Show that S is non-compact.



pf: [Prove by contradiction]
Suppose S is compact
then by previous tutorial
S is compact without boundary
$\Rightarrow$ there is one point $p \in S$ such that
$K(p) > 0$, this is a contradiction to $K(p) \le 0$
$\therefore$ S is non-compact.

2. Let S be a regular surface with $\text{int}(S)$ to be connected.
Let k_1, k_2 be the principle curvatures of S
Suppose $k_1 = k_2 = k \neq 0$ for some constant k for $\forall p \in S$
Show that $\text{int}(S)$ is contained in a sphere with radius $\left|\frac{1}{k}\right|$.

pf: fix any $p \in \text{int}(S)$
then $\forall q \in \text{int}(S)$
$\because$ $\text{int}(S)$ is connected
$\therefore \exists\ \alpha: (-\varepsilon, 1+\varepsilon) \to S$ such that



$$\begin{cases} \alpha(0) = p \\ \alpha(1) = q \\ |\alpha'(s)| = 1 \end{cases}$$

define $c(s) := \alpha(s) + \frac{1}{k}\vec{N}(\alpha(s))$

$$\therefore\ c'(s) = \alpha'(s) + \frac{1}{k}\cdot\frac{d}{dt}\left(N(\alpha(t))\right)\Big|_{t=s} = \alpha'(s) - \frac{1}{k}S_{\alpha(s)}(\alpha'(s))$$

$$= \alpha'(s) - \frac{1}{k}\cdot k\alpha'(s) = \alpha'(s) - \alpha'(s) = 0$$

since $S_p(\vec{v}) = k\vec{v}$ for $\forall p \in S$, $\forall \vec{v} \in T_pS$.

$\therefore$ c is a fixed point and $c = p + \frac{1}{k}\vec{N}(p)$

$$\therefore\ |\alpha(s) - c| = \left|\frac{1}{k}\vec{N}\right| = \left|\frac{1}{k}\right|$$

$$\therefore\ |q - c| = |p - c| = \left|\frac{1}{k}\right| \text{ for } \forall q \in \text{int}(S)$$

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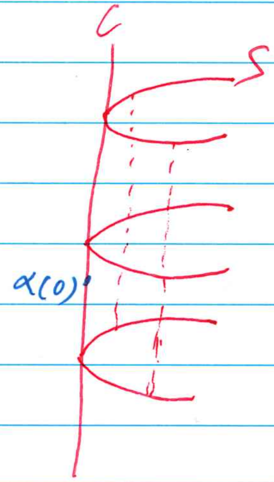
3. If a surface S contains a straight line $L \subseteq S$
Show that $K \le 0$ at every point on L.

pf: let α a arc-length parametrization of L
and $\alpha(0) \in L$



$\therefore \alpha'(s) \equiv \vec{v}$ for some fixed unit vector $\vec{v}$
and $\alpha(s) - \alpha(0) = s\vec{v}$

$\because \alpha'(s) \in T_{\alpha(s)}S$ ($L \subseteq S$)

$\therefore \langle \alpha'(s), N(\alpha(s)) \rangle \equiv 0$

$\therefore \langle \vec{v}, N(\alpha(s)) \rangle \equiv 0$

$\therefore \langle \alpha(s) - \alpha(0), N(\alpha(s)) \rangle$
$= s \langle \vec{v}, N(\alpha(s)) \rangle \equiv 0$

$\therefore \frac{d}{ds} \langle \alpha(s) - \alpha(0), N(\alpha(s)) \rangle \equiv 0$

$\therefore \langle \alpha'(s), N(\alpha(s)) \rangle + \langle \alpha(s) - \alpha(0), \frac{d}{ds} N(\alpha(s)) \rangle \equiv 0$

$\Rightarrow 0 + \langle s\vec{v}, -S_{\alpha(s)}(\alpha'(s)) \rangle \equiv 0$

$\Rightarrow \langle \vec{v}, S_{\alpha(s)}(\vec{v}) \rangle \equiv 0, \ \forall s \neq 0$

$\therefore$ let k_1, k_2 be the principle curvatures of S at $\alpha(s)$ and $k_1 \le k_2$
then $k_1 \le \langle \vec{v}, S_{\alpha(s)}(\vec{v}) \rangle \le k_2$ ($|\vec{v}| = 1$)

$\therefore k_1 \le 0 \le k_2$

$\therefore K(\alpha(s)) = k_1 \cdot k_2 \le 0$ for $\forall s \neq 0$

$\because L$ is a straight line

$\therefore$ let $\beta(s) = \alpha(s-1)$
similarly, $K(\beta(s)) \le 0$ for $\forall s \neq 0$

$\therefore K(\beta(1)) \le 0$
$= K(\alpha(0)) \le 0$

$\therefore K \le 0$ at every point on L

• Remark: let k_1, k_2 be principle curvatures of S at p and $k_1 \le k_2$ and $\vec{v}_1, \vec{v}_2 \in T_pS$ be the corresponding eigenvectors and $|\vec{v}_1| = |\vec{v}_2| = 1$

then $\forall \vec{v} \in T_pS$ with $|\vec{v}| = 1$

$$k_1 \le \langle S_p(\vec{v}), \vec{v} \rangle \le k_2$$

Pf:

$$\langle S_p(\vec{v}), \vec{v} \rangle$$

$$= \langle S_p(\cos\theta\, \vec{v}_1 + \sin\theta\, \vec{v}_2), \cos\theta\, \vec{v}_1 + \sin\theta\, \vec{v}_2 \rangle$$ since $\{\vec{v}_1, \vec{v}_2\}$ is an orthonormal basis of T_pS, $|\vec{v}| = 1$

$$= \langle \cos\theta\, k_1 \vec{v}_1 + \sin\theta\, k_2 \vec{v}_2, \cos\theta\, \vec{v}_1 + \sin\theta\, \vec{v}_2 \rangle$$

$$= \cos^2\theta\, k_1 + \sin^2\theta\, k_2$$

$\therefore$ $\langle S_p(\vec{v}), \vec{v} \rangle \le \cos^2\theta\, k_2 + \sin^2\theta\, k_2 = k_2$

$\langle S_p(\vec{v}), \vec{v} \rangle \ge \cos^2\theta\, k_1 + \sin^2\theta\, k_1 = k_1$

$\therefore$ $k_1 \le \langle S_p(\vec{v}), \vec{v} \rangle \le k_2$