Tutorial 5.

1. Let S be a regular surface, $p \in \text{int}(S)$, $X: U \to S$ is a parametrization around p.
Show that there is a reparametrization around p such that

$$\tilde{g}\big|_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{h}\big|_p = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \text{ for some } k_1, k_2 \in R$$

where k_1, k_2 are called the principle curvatures of S.

pf: let S_p be the shape operator at p.

$\because$ $S_p : T_pS \to T_pS$ is linear, T_pS can be regarded as R^2

$\therefore \exists$ $\vec{v}_1, \vec{v}_2 \in T_pS$ such that

$$\begin{cases} \|\vec{v}_1\| = \|\vec{v}_2\| = 1 \\ \langle v_1, v_2 \rangle = 0 \\ S_p(v_1) = k_1 v_1, \quad S_p(v_2) = k_2 v_2 \quad \text{for some } k_1, k_2 \end{cases}$$

and we can assume $\langle X_u \times X_v, v_1 \times v_2 \rangle > 0$ [the same orientation]

- $\therefore \exists$ a non-singular $\underbrace{\text{matrix}}_{2\times 2}$ A with $\det(A) > 0$ such that

$$A \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix} = \begin{bmatrix} X_u \\ X_v \end{bmatrix}$$

- let $(u_0, v_0) \in U$ such that $X(u_0, v_0) = p$

define $\varphi: R^2 \to U$ by

$$\varphi(\tilde{u}, \tilde{v}) = (A^{-1})^T \left(\begin{bmatrix} \tilde{u} - u_0 \\ \tilde{v} - v_0 \end{bmatrix} \right) + (u_0, v_0)^T$$

$\therefore \varphi$ is a local diffeomorphism around (u_0, v_0)

and $\varphi(u_0, v_0) = (u_0, v_0)$, $\det(d\varphi(u_0, v_0)) > 0$

- Define $Y(\tilde{u}, \tilde{v}) = X \circ \varphi : U \to S$

then $\begin{cases} \partial_{\tilde{u}} Y = \partial_u X \cdot \frac{\partial u}{\partial \tilde{u}} + \partial_v X \cdot \frac{\partial v}{\partial \tilde{u}} \\ \partial_{\tilde{v}} Y = \partial_u X \cdot \frac{\partial u}{\partial \tilde{v}} + \partial_v X \frac{\partial v}{\partial \tilde{v}} \end{cases}$

$$\therefore \begin{bmatrix} \partial_{\tilde{u}} Y \\ \partial_{\tilde{v}} Y \end{bmatrix} = (d\varphi)^T \begin{bmatrix} \partial_u X \\ \partial_v X \end{bmatrix} = A^{-1} \begin{bmatrix} X_u \\ X_v \end{bmatrix} = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix}$$

$\therefore$ $\tilde{g}(p) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ since $\{\vec{v}_1, \vec{v}_2\}$ is an orthonormal basis

$\tilde{h}(p) = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$ since $S_p(\vec{v}_1) = k_1 \vec{v}_1$, $S_p(\vec{v}_2) = k_2 \vec{v}_2$

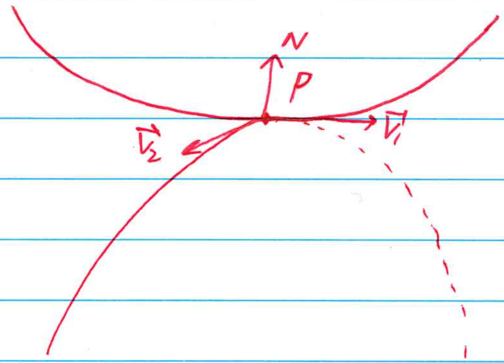
2. show that $K(p) = k_1 \cdot k_2$, $H(p) = \frac{1}{2}(k_1 + k_2)$

pf: $K(p) = \frac{\det(h)}{\det(g)} = \frac{k_1 \cdot k_2}{1} = k_1 \cdot k_2$

$H(p) = \frac{1}{2} \cdot \frac{k_1 + k_2}{1} = \frac{1}{2}(k_1 + k_2)$

3. let S be a regular surface
Suppose $H(p) = 0$
show that $K(p) \leq 0$

pf: choose a parametrization around p
such that $g(p) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $h(p) = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$

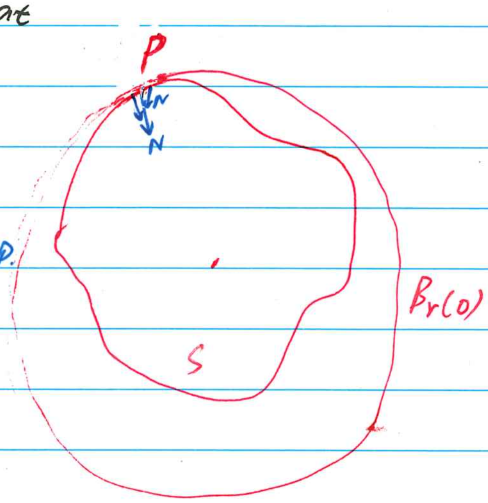


then $H(p) = 0 \Rightarrow k_1 + k_2 = 0 \Rightarrow k_1 = -k_2$

$\therefore K(p) = k_1 \cdot k_2$
$= -k_1 \cdot k_1 = -k_1^2 \leq 0$

4. let S be a regular compact surface without boundary.
Show that there is a point $p \in S$ such that
$K(p) > 0$

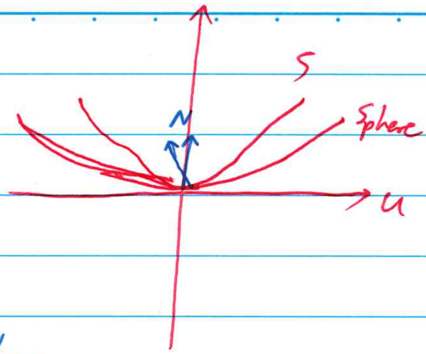
pf: 1° since S is compact
$\therefore \exists\, r > 0$ such that
S is inside $B_r(0)$ and $B_r(0)$ is tangent to S at p.
choose $\vec{N}$ to be the unit normal vector
2° choose a reparametrization around p
such that $g = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $h = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$



3°. After rotation and translation

$P=(0,0,0)$, $\partial u|_p=(1,0,0)$, $\partial v|_p=(0,1,0)$

$N(P)=(0,0,1)$



and S is a graph over $\mathbb{R}^2$ around p

let $(u,v,F(u,v))$ denote S

$(u,v,\ G(u,v)=r-\sqrt{r^2-(u^2+v^2)})$ denote the sphere.

$\therefore \begin{cases} F(0,0)=G(0,0)=0, & \partial_u F(0,0)=\partial_v F(0,0)=\partial_u G(0,0)=\partial_v G(0,0)=0 \\ F(u,v)\geq G(u,v) \end{cases}$

$\Rightarrow \partial_u F(0,0)\geq \partial_u G(0,0)$ and $\partial_u^2 F(0,0)\geq \partial_u^2 G(0,0)$

use $f''(x)=\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2}$

• $N(u,v)=(f,g,h)$ and $h(0,0)=1$

$\therefore \langle N, \partial_u S\rangle=0$

$\Rightarrow f+h\,\partial_u F=0$

$\Rightarrow f_u+h_u\partial_u F+h\,\partial_u^2 F=0$

$\therefore \langle \partial_u N, \partial_u S\rangle|_{(0,0)}$

$=f_u+h_u\partial_u F$

$=-h\,\partial_u^2 F$

$\leq -h\,\partial_u^2 G$ since $h>0$, $\partial_u^2 F\geq \partial_u^2 G$

<0 since $\begin{cases} h=1 \text{ at } P \\ \partial_u^2 G>0 \text{ at } P \end{cases}$ $(G=r-\sqrt{r^2-(u^2+v^2)})$

$\therefore \langle k_1\partial_u S, \partial_u S\rangle<0$ at P

$\therefore k_1<0$ at P

Similarly, $k_2<0$ at P

$\therefore K(P)=k_1\cdot k_2>0$ at P.