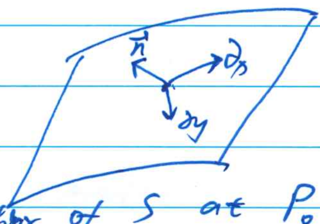
Tutorial 4.

1. show that the equation of the tangent plane of a surface which is the graph of a differentiable function $z = f(x,y)$, at point $P_0 = (x_0, y_0, f(x_0,y_0))$ is given by

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

pf: let $X: (x,y) \to (x,y,f(x,y))$ be a parametrization of S



$\therefore \partial_x X = (1, 0, f_x)$

$\partial_y X = (0, 1, f_y)$

$\therefore \vec{n} = (\partial_x X) \times (\partial_y X) = (-f_x, -f_y, 1)$ is a normal vector of S at P_0

$\therefore$ the tangent space is

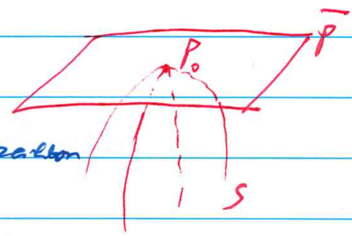
$\langle (x,y,z) - (x_0, y_0, f(x_0,y_0)), \vec{n} \rangle = 0$

$\therefore \quad z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$.

2. let S be a regular surface. let $\bar{P}$ be a plane.

suppose $S \cap \bar{P} = \{P_0\}$ and $P_0 \in int(S)$, $P_0 \in int(\bar{P})$

Show $\bar{P}$ is tangent to S at P_0



Pf: $\because P_0 \in int(S)$

$\therefore \exists$ an open set $U \subseteq R^2$, $X: U \to S$ a parametrization

such that $\begin{cases} X(0,0) = P_0, \ (0,0) \in U \\ \text{and} \quad B_\varepsilon(0) \subseteq U \text{ for some } \varepsilon > 0 \end{cases}$

let $\vec{n}$ be a unit normal vector of $\bar{P}$ at P_0

define $\quad f(u,v) = \langle X(u,v) - P_0, \vec{n} \rangle : U \to R$

$\because S \cap \bar{P} = \{P_0\}$, $\quad P_0 \in int(\bar{P})$

$\therefore \quad f(u,v) \neq 0$ for $\forall (u,v) \in U \setminus (0,0)$, and $f(0,0) = 0$

[claim: $f_u(0,0) = f_v(0,0) = 0$]

Suppose $f_u(0,0) > 0$, then $\exists \ \varepsilon_1 \in (0, \varepsilon)$ such that

$f(\varepsilon_1, 0) > 0$, $f(-\varepsilon_1, 0) < 0$

$\therefore \exists \ \varepsilon_2 > 0$ such that $\quad f(\varepsilon_1, \varepsilon_2) > 0$, $f(-\varepsilon_1, \varepsilon_2) < 0$

$\because f(\cdot, \varepsilon_2): R \to R$ is smooth, $f(\varepsilon_1, \varepsilon_2) > 0$, $f(-\varepsilon_1, \varepsilon_2) < 0$

$\therefore \exists \ t_0 \in (-\varepsilon_1, \varepsilon_2)$ such that $f(t_0, \varepsilon_2) = 0$, contradiction

$\therefore f_u(0,0) = 0$, similarly, $f_v(0,0) = 0$ $\quad \therefore \langle \partial_u X(0,0), \vec{n} \rangle = \langle \partial_v X(0,0), \vec{n} \rangle = 0$

3、[Reparametrization]

Let S be a regular surface. $P \in \mathrm{int}(S)$

Let U be an open set in R^2, and $X: U \to S$ be a parametrization around P. W.L.O.G, we can assume $(0,0) \in U$ and $X(0,0)=p$

Show that there is a reparametrization of S around P such that $\tilde{g}_{ij}(P) = \delta_{ij}$ where $\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

Pf: let $\tilde{U}$ be an open set in R^2 around $(0,0)$

and $\varphi: U \to \tilde{U}$ be a diffeomorphism with $\det(d\varphi) > 0$ everywhere.

let $Y = X \circ (\varphi^{-1}) : \tilde{U} \to S$ be a reparametrization.

$$\therefore \begin{aligned} \partial_{u'} Y &= \partial_u X \cdot \frac{\partial u}{\partial u'} + \partial_v X \cdot \frac{\partial v}{\partial u'} \\ \partial_{v'} Y &= \partial_u X \cdot \frac{\partial v}{\partial v'} + \partial_v X \cdot \frac{\partial v}{\partial v'} \end{aligned}$$

$$\therefore \begin{cases} g_{u'u'} = \langle \partial_{u'} Y, \partial_{u'} Y \rangle = \left(\frac{\partial u}{\partial u'}\right)^2 g_{uu} + 2 \frac{\partial u}{\partial u'} \frac{\partial v}{\partial u'} g_{uv} + \left(\frac{\partial v}{\partial u'}\right)^2 g_{vv} \\ \cdots \end{cases}$$

$$\therefore \tilde{g} = \begin{bmatrix} \frac{\partial u}{\partial u'} & \frac{\partial v}{\partial u'} \\ \frac{\partial u}{\partial v'} & \frac{\partial v}{\partial v'} \end{bmatrix} \begin{bmatrix} g_{uu} & g_{uv} \\ g_{uv} & g_{vv} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial u'} & \frac{\partial u}{\partial v'} \\ \frac{\partial v}{\partial u'} & \frac{\partial v}{\partial v'} \end{bmatrix}$$

$$= (d\varphi^{-1})^T g \, (d\varphi^{-1})$$

$\because g$ is a matrix with $\det(g) > 0$

$\therefore \exists$ such $d\varphi^{-1}$ as a change of coordinate such that

$$\tilde{g} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$(I = A^T g A \Leftrightarrow g = (A^{-1})^T A^{-1}$

choose $\vec{v}_1, \vec{v}_2 \in R^2$ such that $g_{ij} = \langle \vec{v}_i, \vec{v}_j \rangle$, let $(A^{-1})^T = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix}$

check $g = (A^{-1})^T A^{-1}$

$\therefore \varphi^{-1}$ exists and $(d\varphi^{-1})^T g (d\varphi^{-1})(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- Remark: we can only guarantee that $\tilde{g}(P) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

  because we can only choose $d\varphi^{-1}|_{(0,0)}$ to satisfy $\tilde{g}(P) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- Actually, we can choose a Coordinate such that

$$\begin{cases} \tilde{g}_{ij}(P) = \delta_{ij} \\ \partial_k \tilde{g}_{ij}(P) = 0 \quad \text{for } \forall i,j,k \end{cases}$$

  Such coordinate is called normal coordinate.

  So if we want to define something which is independent of the choice of coordinate, then it must involves 2nd-derivatives of g.