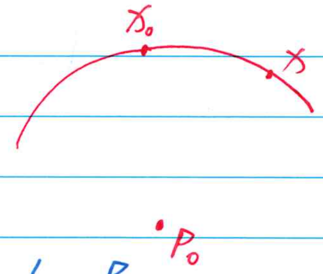
Tutorial 3

1. If all normal lines to a connected surface $S$ passes through a fixed point $P_0 \in R^3$, show that $S$ is contained in a sphere.




pf: • Choose any $x_0 \in S$

then $\forall\, x \in S$, since $S$ is connected

$\exists$ a regular curve $\alpha: [0,1] \to S$ such that

$\alpha(0) = x_0, \quad \alpha(1) = x$

• $\because$ the normal line of $S$ at $\alpha(t)$ passes through $P_0$

$\therefore \langle \alpha'(t), \alpha(t) - P_0 \rangle \equiv 0$

$\therefore \langle \alpha(t) - P_0, \alpha(t) - P_0 \rangle' \equiv 0$

$\therefore \langle \alpha(t) - P_0, \alpha(t) - P_0 \rangle$ is a constant

$\therefore \langle \alpha(1) - P_0, \alpha(1) - P_0 \rangle = \langle \alpha(0) - P_0, \alpha(0) - P_0 \rangle$

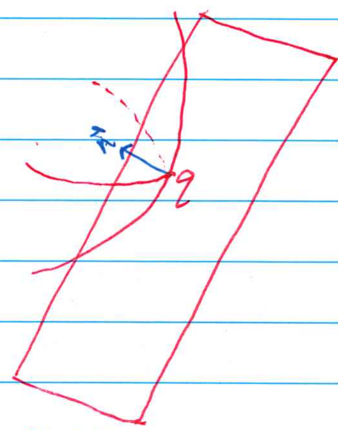
$\therefore \|x - P_0\| = \|x_0 - P_0\|, \quad \forall\, x \in S$

$\therefore S$ is contained in a sphere centered at $P_0$.

2. Let $S \subseteq R^3$ be a surface.

Suppose $P \subseteq R^3$ is a plane such that $S$ lies on one side of $P$, show that $T_qP = T_qS$ at all $q \in \operatorname{int}(P) \cap \operatorname{int}(S)$

where $\operatorname{int}(S)$ means the interior of $S$.




pf: • Let $\vec{n}$ be a unit normal vector of $P$ at $q$ which points to $S$.

Let $X: (u,v) \to S$ be its parametrization

and $X(u_0, v_0) = q$

$\because q \in \operatorname{int}(S)$

$\therefore \exists\, \varepsilon_0 > 0$ such that

$\begin{cases} \alpha(t) = X(u_0 + t, v_0) \\ \beta(t) = X(u_0, v_0 + t) \end{cases}$ are two curves on $S$ for $t \in (-\varepsilon_0, \varepsilon_0)$

• Let $f(t) = \langle \alpha(t) - q, \vec{n} \rangle$

$\therefore f(t) \geq 0$ since $S$ lies on one side of $P$.

$\because$ $f(0)=0$, $f(t)\geq 0$ for $t\in(-\varepsilon_0,\varepsilon_0)$

$\therefore$ $f'(0)=0$

$\therefore$ $\langle \alpha'(0), \vec{n}\rangle = 0$

$\therefore$ $\partial_u X(u_0,v_0) \perp \vec{n}$

similarly, $\partial_v X(u_0,v_0)\perp\vec{n}$

$\therefore$ $T_qS \perp \vec{n}$

$\therefore$ $T_qS = T_qP$ since $q\in int(P)$, $\vec{n}\perp T_qP$.

3. Let $S$ be a closed surface [compact without boundary]

Let $\vec{a}$ be any unit vector in $R^3$

Show that there exists a point on $S$ whose normal line is parallel to $\vec{a}$.

pf: $\because$ $S$ is compact

$\therefore$ $\exists$ a complete plane [with no boundary] $\overline{P}$ such that

$S$ lies on one side of $\overline{P}$, $\overline{P}\cap S=\phi$, $\vec{a}$ points to $S$, $\vec{a}\perp P$

• So move $\overline{P}$ in $\vec{a}$ direction

then stop when $\overline{P}$ touches $S$ the first time

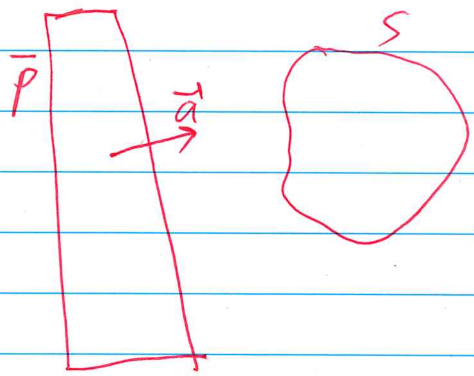
• let $q\in S\cap\overline{P}$

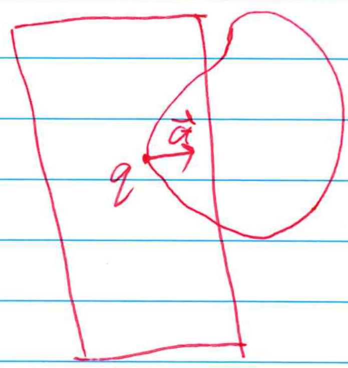
$\therefore$ $q\in int(S)\cap int(\overline{P})$ since $\partial S=\phi$

and $S$ lies on one side of $\overline{P}$

$\therefore$ By question 2,

$\vec{a}\perp T_qS$ #.



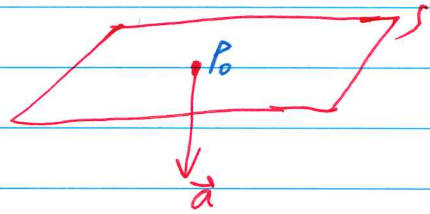



4. Let $S$ be a connected surface.

Let $a \in \mathbb{R}^3$ be a unit vector

Suppose all the normal lines of $S$ are parallel to $\vec{a}$.

Show that $S$ is contained in a plane




pf: fix any $P_0 \in S$

then $\forall P \in S$, since $S$ is connected

$\exists$ a regular curve $\alpha : [0,1] \to S$ such that

$\alpha(0) = P_0, \quad \alpha(1) = P$

let $f(t) = \langle \alpha(t) - P_0, \vec{a} \rangle$

$\therefore f(0) = 0, \quad f'(t) = \langle \alpha'(t), \vec{a} \rangle \equiv 0$ since $\vec{a} \perp T_{\alpha(t)} S$

$\therefore f(t) \equiv 0$

i.e. $\langle P - P_0, \vec{a} \rangle = 0$

$\therefore S$ is contained in a plane passing through $P_0$