Tutorial 2:

1. Let $\alpha: I \to R^2$ be a regular curve such that all the normal lines pass through a fixed point $P \in R^2$

Prove that the trace $\alpha(I)$ is contained in a circle of some radius $r > 0$ centered at P.

pf: $\because$ α is regular

$\therefore$ $\alpha'(s) \neq 0$

$\because$ $\alpha'(s) \perp$ the normal line of the curve at $\alpha(s)$

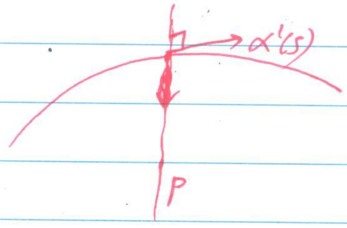
the normal line of the curve at $\alpha(s)$ passes $\alpha(s)$ and P

$\therefore$ $\alpha'(s) \perp (\alpha(s) - P)$

$\therefore$ $\langle \alpha'(s), \alpha(s) - P \rangle \equiv 0$

$\therefore$ $\langle \alpha(s) - P, \alpha(s) - P \rangle' \equiv 0$

$\therefore$ $|\alpha(s) - P|$ is a constant.



2. Let $\alpha: I \to R^2$ be a regular curve such that all the tangent lines pass through a fixed point $P \in R^2$.

Prove that the trace $\alpha(I)$ is contained in a straight line passing through P.

pf: $\because$ α is regular

$\therefore$ we can reparametrize α by arc length

$\therefore$ $\langle \alpha'(s), \alpha'(s) \rangle \equiv 1 \Rightarrow \langle \alpha'(s), \alpha''(s) \rangle \equiv 0$

$\because$ the tangent line of the curve passes through P

$\therefore$ $\alpha(s) - P = f(s) \cdot \alpha'(s)$ for some function f

$\therefore$ $\alpha'(s) = f'(s)\alpha'(s) + f(s)\alpha''(s)$

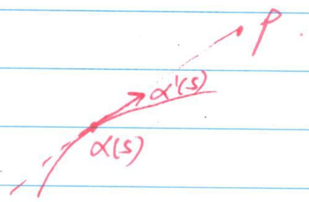
$\langle \alpha'(s), \alpha'(s) \rangle = \langle f'(s)\alpha'(s) + f(s)\alpha''(s), \alpha'(s) \rangle$

$\therefore$ $1 \equiv f'(s)$

$\therefore$ $f(s) = s + s_0$ for some s_0

$\therefore$ $\alpha'(s) = 1 \cdot \alpha'(s) + (s + s_0)\alpha''(s)$

$\therefore$ $(s + s_0)\alpha''(s) \equiv 0$ for $\forall s \in I$



if $-s_0 \notin I \Rightarrow \alpha''(s) \equiv 0 \Rightarrow \alpha'(s)$ is a constant $\Rightarrow \alpha(s)$ is a straight line passing through P.

if $-S_0 \in I$

then $\alpha''(s) = 0$ for $I \setminus \{-S_0\}$

$\therefore \alpha'(s) = \int_{S_0}^{S} \alpha''(\tau) d\tau + \alpha'(S_0) = \alpha'(S_0) \quad \forall s \in I \setminus \{-S_0\}$.

$\therefore \alpha'(s)$ is a constant

$\therefore \alpha(s)$ is a straight line which passes through p

3. let $\alpha: I \to R^2$ be a regular closed curve parametrized by arc length

define $e_1(s) = \alpha'(s) = (x'(s), y'(s))$

$e_2(s) = (-y'(s), x'(s))$ rotate $e_1(s)$ by $\frac{\pi}{2}$ counterclockwisely

define $k(s) = \langle e_1'(s), e_2(s) \rangle$ which is called the signed curvature

then $e_1'(s) = k(s) e_2(s)$

Show that $\int_0^l k(s) ds \in 2\pi \cdot \mathbb{Z}$

pf: $\alpha: [0, l] \to R^2$ is a regular closed curve p.b.a.l

$\therefore e_1(s) = (\cos\varphi(s), \sin\varphi(s))$ for some $\varphi(s)$

$e_2(s) = (-\sin\varphi(s), \cos\varphi(s))$

$e_1'(s) = (-\sin\varphi(s), \cos\varphi(s)) \cdot \varphi'(s)$

$\therefore k(s) = \langle e_1'(s), e_2 \rangle$

$= \varphi'(s)$

$\therefore \int_0^l k(s) ds = \int_0^l \varphi'(s) ds = \varphi(s) - \varphi(0) \in 2\pi n$ for some integer n

since $\varphi(s)$ is the angle of $\alpha'(s)$