Tutorial 11.

1. Let $\Sigma \subseteq R^3$ be a closed (compact without boundary) connected surface which is not homeomorphic to a sphere. Prove that there are points on Σ where the Gauss curvature K is positive, negative and zero respectively.

Pf: $\because$ Σ is compact without boundary

$\therefore$ By previous result, there exists at least one point p ($|p|$ is a maximum) such that $K(p) > 0$

$\because$ Σ is not homeomorphic to a sphere

$\therefore$ $\chi(\Sigma) \leq 0$

$\therefore$ By the Gauss - Bonnet Theorem,

$$\iint_\Sigma K \, dA = 2\pi \chi(\Sigma) \leq 0$$

$\because$ $K(p) > 0$, K is continuous on Σ

$\therefore$ $\exists$ another point q such that

$K(q) < 0$

$\because$ Σ is connected

$\therefore$ $\exists$ a curve $\alpha : [0,1] \to \Sigma$ such that

$\alpha(0) = p$, $\alpha(1) = q$

$\because$ $K(0) > 0$, $K(1) < 0$

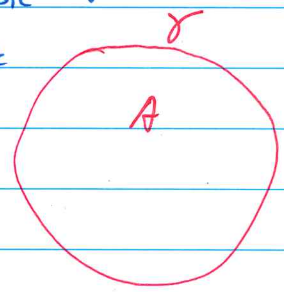
$\therefore$ $\exists$ a t_0 such that

$K(\alpha(t_0)) = 0$

2. Let $\Sigma = (\mathbb{R}^2, g)$ be an abstract surface with a Riemannian metric. Suppose the Gauss curvature $K < 0$ on Σ, then show that there is no simple closed geodesic. Futhermore, show that there is no closed geodesic.

pf: (a) Suppose there is a simple closed geodesic γ

$\therefore$ the domain A bounded by γ is homeomorphic to a disk



$\therefore \quad \chi(A) = 1$

$\therefore$ By Gauss-Bonnet Theorem

$$\iint_A K \, dA = 2\pi \chi(A) = 2\pi$$

But $K < 0$ on Σ everywhere.

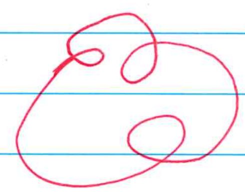
$\therefore$ we have a contradiction

$\therefore$ there is no simple closed geodesic on Σ

(b) Suppose there is a closed geodesic $\beta : [0, l] \to \Sigma$

$\therefore$ there is a least $t_0 \in (0, l)$ such that

$\beta(t_0) = \beta(t_1)$ for some $t_1 \in [0, t_0)$

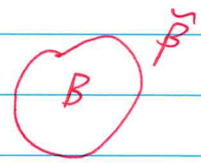


$\therefore \tilde{\beta} : [t_1, t_0] \to \Sigma$ by $\tilde{\beta}(t) = \beta(t)$

is a simple closed curve and $k_g(t) \equiv 0$

But $\tilde{\beta}'(t_1)$ may be different with $\tilde{\beta}'(t_0)$

$\therefore$ By Gauss-Bonnet Theorem



$$\iint_B K \, dA + \theta_1 = 2\pi \chi(B) = 2\pi$$

where θ_1 is the exterior angle at $\tilde{\beta}(t_0)$

$\because K < 0$

$\therefore \quad 2\pi = \iint_B K \, dA + \theta_1 < 0 + \theta_1 = \theta_1 \leq \pi$

which is a contradiction since $\theta_1 \in [-\pi, \pi]$

$\therefore$ Such curve doesn't exist.