Tutorial 10.

(1) Let $\alpha(s)$ be an arc-length parametrized curve lying on the unit sphere. Let k be the curvature of α as a space curve in R^3. Prove that $k \geq 1$ everywhere and if $k \equiv 1$, then α is a geodesic on S^2.

Pf: $\alpha'' = k_g \vec{n} + k_n \vec{N}$

$\therefore |k| = \sqrt{k_g^2 + k_n^2}$

$\because |k_n| = |\langle \alpha'', \vec{N} \rangle| = |\langle \alpha', S_p(\alpha') \rangle| = |\langle \alpha', \alpha' \rangle| = 1$

$S_p(\alpha') = \pm\alpha'$ since α lies on a sphere.

$\therefore |k| = \sqrt{k_g^2 + k_n^2} \geq \sqrt{k_g^2 + 1} \geq 1$

and $|k| = 1 \Leftrightarrow k_g = 0$

$\therefore$ if $k \equiv 1$ then α is a geodesic

(2) Let T be a torus of revolution with parametrization

$$X(u,v) = ((r\cos v + a)\cos u, (r\cos v + a)\sin u, r\sin v)$$

(a) Compute the geodesic curvature of the upper parallel $v = \frac{\pi}{2}$

(b) If a geodesic is tangent to the parallel $v = \frac{\pi}{2}$, show that it is entirely contained in the region of T given by

$$-\frac{\pi}{2} \leq v \leq \frac{\pi}{2}$$

(c) A geodesic without end point that intersects the parallel $v = 0$ under an angle θ_0 ($0 < \theta_0 < \frac{\pi}{2}$) also intersects the parallel $v = \pi$ if and only if

$$\cos\theta_0 < \frac{a-r}{a+r}$$

Ans: (a) $v = \frac{\pi}{2} \Rightarrow \alpha(u) = (a\cos u, a\sin u, r)$

Reparametrize α by arc length, then

$$\alpha(s) = (a\cos\frac{s}{a}, a\sin\frac{s}{a}, r)$$

$\therefore \alpha''(s) = -\frac{1}{a}(\cos\frac{s}{a}, \sin\frac{s}{a}, 0)$

$\because \alpha'' \perp \vec{N} = (0,0,1)$

$\therefore |k_g| = |\alpha''(s)| = \frac{1}{a}$

(b) Since $r(s) \cdot \sin\theta(s)$ is a constant for a geodesic on a revolution surface and this geodesic is tangent to $v = \frac{\pi}{2}$

$\therefore$ $r(s) \cdot \sin\theta(s) = a \cdot 1 = a$

$\therefore$ $r(s) \cdot 1 \geq r(s) \cdot \sin\theta(s) = a$

$\therefore$ $r(s) \geq a$

$\therefore$ $r\cos v + a \geq a$

$\therefore$ $\cos v \geq 0$

$\therefore$ $v \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

which means it will never cross $v = -\frac{\pi}{2}$ and $v = \frac{\pi}{2}$

(c) "$\Rightarrow$" $\because$ it is a geodesic

$\therefore$ $r(s) \cdot \sin\theta(s)$ is a constant

$\because$ it intersects $v = 0$ with angle θ_0

$\therefore$ $r(s) \cdot \sin\theta(s) = (r+a) \cdot \sin(\frac{\pi}{2} - \theta_0) = (r+a)\cos(\theta_0)$

$\because$ it also intersects $v = \pi$

$\therefore$ $(r+a)\cos\theta_0 = r(s) \cdot \sin\theta(s) \leq (a-r) \cdot 1$

$\therefore$ $\cos\theta_0 \leq \frac{a+r}{a-r}$

and "=" holds if and only if $r = a-r$, $\theta = \frac{\pi}{2}$
which is a contradiction since $v = \pi$ is the only geodesic which has $r = a-r$, $\theta = \frac{\pi}{2}$

$\therefore$ $\cos\theta_0 < \frac{a+r}{a-r}$

"$\Leftarrow$" Given $\cos\theta_0 < \frac{a+r}{a-r}$

W.L.O.G, we can assume $\beta(0)$ lies on $v = 0$ and $v'(0) > 0$

[Prove by contradiction]

Suppose as s increases, $\beta(s)$ does not intersect $v = \pi$

Case I: $\exists$ some $N_0 > 0$ such that

$V(s) > 0$ on $(0, N_0)$

$V(0) = V(N_0) = 0$

then $V(s)$ has a maximum $V(s_0) \in (0, \pi)$ for some $s_0 \in (0, N_0)$

$\because$ $\beta(s)$ does not intersect $V = \pi$

$\therefore$ $r(s_0) > a - r$

$\because$ $V(s_0)$ is a maximum $\quad \therefore V'(s_0) = 0$

$\therefore$ $\alpha'(s_0) = X_u \cdot u'(s_0) + X_v \cdot V'(s_0) = X_u \cdot u'(s_0)$

$\therefore$ $\theta(s_0) = \frac{\pi}{2}$

$\therefore$ $r(s_0) \cdot \sin\theta(s_0) > (a-r) \cdot \sin\frac{\pi}{2} = a - r$

$= r(0) \cdot \sin\theta(0)$

$= (a+r)\sin(\frac{\pi}{2} - \theta_0)$

$= (a+r)\cos\theta_0$

$\therefore$ $\cos\theta_0 > \frac{a-r}{a+r}$ which is a contradiction to $\cos\theta_0 < \frac{a-r}{a+r}$

Case II: For any $s \in (0, \infty)$, $V(s) > 0$ and $V(s)$ has no maximum

$\therefore$ Let $V_0 = \sup\limits_{s \in (0, +\infty)} V(s)$ and $V(s) \to V_0$ as $s \to \infty$

$\therefore$ $V'(s) \to 0$ as $s \to \infty$

$\therefore$ $\cos\theta(s) = \langle \alpha', e_2 \rangle = \langle \alpha', \frac{X_v}{|X_v|} \rangle = \langle X_u \cdot u' + X_v \cdot v', \frac{X_v}{|X_v|} \rangle$

$= |X_v| \cdot V' \to 0$ since $\langle X_u, X_v \rangle = 0$, $|X_v|$ is bounded.

$\therefore$ $\theta(s) \to \frac{\pi}{2}$

$\therefore$ $r(s) \cdot \sin\theta(s) \to \phi(V_0) \cdot \sin\frac{\pi}{2} = \phi(V_0)$ as $s \to \infty$

$= r(0) \cdot \sin\theta(0)$

$= (a+r)\cos\theta_0$

$\therefore$ $\cos\theta_0 = \frac{\phi(V_0)}{a+r} \geq \frac{a-r}{a+r}$ which is also a contradiction to $\cos\theta_0 < \frac{a-r}{a+r}$

$\therefore$ $\beta(s)$ must intersects $V = \pi$ if $\cos\theta_0 < \frac{a-r}{a+r}$

(Actually Case II contains $\cos\theta_0 = \frac{a-r}{a+r}$, then as $s \to \infty$, $\beta(s)$ will converge to $V = \pi$)