Tutorial 1, 6 Sep.

1. Reparametrize the following curves by arc length

(a) $\alpha: R \to R^2 : t \mapsto (r\cos kt,\ r\sin kt)$, $k, r > 0$

(b) $\alpha: R \to R^3 : t \mapsto (a\cos t,\ a\sin t,\ bt)$

Solution: (a) $\alpha'(t) = (-kr\sin kt,\ kr\cos kt)$

$|\alpha'(t)| = kr$

$\therefore\ s(t) = \int_0^t rk\, dt = rkt$

$\therefore\ t(s) = \frac{1}{rk} s$

$\therefore\ \beta(s) = \alpha(t(s)) = (r\cos(\frac{s}{r}),\ r\sin(\frac{s}{r}))$

(b) $\alpha'(t) = (-a\sin t,\ a\cos t,\ b)$

$|\alpha'(t)| = \sqrt{a^2+b^2}$

$\therefore\ s(t) = \int_0^t \sqrt{a^2+b^2}\, dt = t\sqrt{a^2+b^2}$

$\therefore\ t(s) = \frac{s}{\sqrt{a^2+b^2}}$

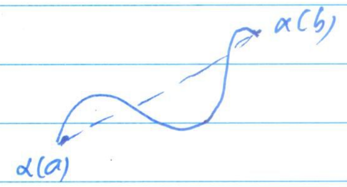
$\therefore\ \beta(s) = \alpha(t(s)) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}},\ a\sin\frac{s}{\sqrt{a^2+b^2}},\ \frac{bs}{\sqrt{a^2+b^2}}\right)$

2. Let $\alpha: I \to R^3$ be a curve and $[a,b] \subseteq I$, prove that

$$|\alpha(a) - \alpha(b)| \leq L_a^b(\alpha)$$

In other words, straight lines are the shortest curves joining two given points

pf: let $\vec{n} = \frac{\alpha(b) - \alpha(a)}{|\alpha(b) - \alpha(a)|}$



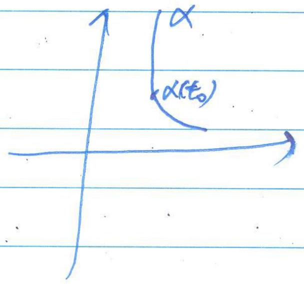
$\therefore\ |\alpha(b) - \alpha(a)| = \int_a^b \langle \alpha'(t), \vec{n} \rangle\, dt$

$\leq \int_a^b |\alpha'(t)| \cdot |\vec{n}|\, dt$

$= \int_a^b |\alpha'(t)|\, dt$

$= L_a^b(\alpha)$

3. Let $\alpha: I \to R^3$ be a curve which does not pass through the origin (i.e. $\alpha(t) \neq 0$ for all $t \in I$). If $\alpha(t_0)$ is the point on the trace of α which is closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$



pf: define $f(t) = |\alpha(t)|^2$

$\therefore f(t) = \langle \alpha(t), \alpha(t) \rangle$

$\therefore f'(t) = 2\langle \alpha'(t), \alpha(t) \rangle$

$\because f(t_0)$ is a local minimum

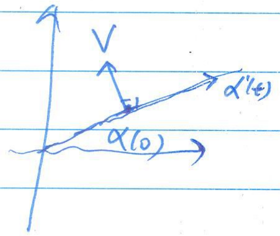
$\therefore f'(t_0) = 0$

$\therefore \langle \alpha'(t_0), \alpha(t_0) \rangle = 0$

4. Let $\alpha: I \to R^3$ be a curve and $V \in R^3$ be a fixed vector. Assume that $\langle \alpha'(t), V \rangle = 0$ for all $t \in I$

and $\langle \alpha(0), V \rangle = 0$

Prove that $\langle \alpha(t), V \rangle = 0$ for all $t \in I$.



pf: Let $f(t) = \langle \alpha(t), V \rangle$

$\therefore f'(t) = \langle \alpha'(t), V \rangle + \langle \alpha(t), 0 \rangle$

$= \langle \alpha'(t), V \rangle = 0$

$\because f(0) = \langle \alpha(0), V \rangle = 0$

$\therefore f \equiv 0$