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<u>Proof of Prop 1</u>

(i) Let $\vec{w} \in T_p(M)$, $\vec{w} = (a\vec{X}_u + b\vec{X}_v)\big|_p$

a, b are numbers.

Let $\alpha(t)$ as in the definition s.t.

$\alpha(t) = \vec{X}(u(t), v(t))$

$\vec{w} = \alpha'(0) = \vec{X}_u u'(0) + \vec{X}_v v'(0) =$

$$\frac{dN}{dt}\Big|_{t=0} = \frac{\partial N}{\partial u} u'(0) + \frac{\partial N}{\partial v} v'(0)$$

$$= a\, \underline{N_u} + b\, \underline{N_v}\big|_p$$

= depends only on w.

$\therefore$ well-defined.

(ii) Need to prove (a) $S_p(\vec{w}) \in T_p(M)$.

Suff. to prove $N_v, N_u \in T_p(M)$

Now: $2\langle N_u, N \rangle = \frac{\partial}{\partial u}\langle N, N \rangle$

$= 0$

$\therefore N_u \perp N$ and $N_u \in T_p(M)$

etc.

(b) S_p is linear. (Why?)

(iii) Let $Y(\xi, \eta)$ be another paramitzation defined on V s.t.:

$$Y = X \circ \phi. \quad \phi : V \to U \text{ is a diffeomorphism.}$$

$N = \text{unit normal vector field}$

$N = N(u, v), \quad u = u(\xi, \eta), \; v = v(\xi, \eta)$

$$\begin{aligned}
\vec{w} \in T_p(M) &= a\vec{X}_u + b\vec{X}_v \\
&= a\left(\vec{Y}_\xi \frac{\partial \xi}{\partial u} + \vec{Y}_\eta \frac{\partial \eta}{\partial u}\right) \\
&\quad + b\left(Y_\xi \frac{\partial \xi}{\partial v} + Y_\eta \frac{\partial \eta}{\partial v}\right) \\
&= \left(a\frac{\partial \xi}{\partial u} + b\frac{\partial \xi}{\partial v}\right)\vec{Y}_\xi \\
&\quad + \left(\cancel{a+}\, a\frac{\partial \eta}{\partial u} + b\frac{\partial \eta}{\partial v}\right)\vec{Y}_\eta.
\end{aligned}$$

In (u, v) coord.

$$S_p(w) = -(a\vec{N}_u + b\vec{N}_v)$$

In (ξ, η) coord

$$\begin{aligned}
S_p(w) = &-(a\xi_u + b\xi_v)N_\xi \\
&-(a\eta_u + b\eta_v)N_\eta.
\end{aligned}$$

But $N_u = N_\xi \xi_u + N_\eta \eta_u$ etc.

Hence $S_p(w)$ gives same vector in both coord.

(iv) Self-adjoint means for $w_1, w_2 \in T_p(M)$

$$\langle S_p(w_1), w_2 \rangle = \langle w_1, S_p(w_2) \rangle.$$

Sufficient to prove

$$\langle S_p(X_u), X_v \rangle = \langle X_u, S_p(X_v) \rangle.$$

(Why?)

$$S_p(X_u) = -N_u.$$

$$L.S. = \langle -N_u, X_v \rangle = -\frac{\partial}{\partial u}\langle N, X_v \rangle + \langle N, X_{vu} \rangle$$

$$= \langle N, X_{vu} \rangle.$$

$$R.S. = \langle X_{uv}, N \rangle.$$

$\because X_{uv} = X_{vu}$ $\therefore$ results follows.

(V) S_p is smooth means:

If $\vec{w}$ is a smooth $\underbrace{\text{vector field}}_{\text{tangential}}$,

then $S_p(\vec{w})$ is a smooth tangential vector field.

Now: Let $\vec{w}$ = smooth, then

$$\vec{w} = a(u,v)\,\vec{X}_u + b(u,v)\,\vec{X}_v.$$

a, b are smooth.

$$S_p(w) = -(a N_u + b N_v).$$

$\because$ N_u, N_v are smooth.

So $S(w)$ is smooth.

Def. 2nd fundamental form II_p at p is the biliner form $\mathrm{II}_p(w_1, w_2)$

$$= \langle S_p(w_1), w_2 \rangle.$$

Prop 2 (Proof). $\mathrm{II}_p(w_1, w_2) = \mathrm{II}_p(w_2, w_1)$

(Why?

To compute the 2nd fundamental form:

Let $\vec{X}: U \to \mathbb{R}^3$ as before.

Let $N = \dfrac{X_u \times X_v}{|X_u \times X_v|}$.

$\because$ II_p is ~~te~~ determined by its values on basis vectors X_u, X_v, we let

$$e = \mathrm{II}_p(X_u, X_u), \quad f = \mathrm{II}_p(X_u, X_v) = \mathrm{II}_p(X_v, X_u)$$

$$g = \mathrm{II}_p(X_v, X_v)$$

Then if $w_1 = a_1 X_u + b_1 X_v$
$w_2 = a_2 X_u + b_2 X_v$, then

$$\mathrm{II}_p(w_1, w_2) = a_1 a_2 e + (a_1 b_2 + b_1 a_2) f + b_1 b_2 g.$$

$$\mathrm{II}_p(w_1, w_1) = a_1^2 e + 2 a_1 b_1 f + b_1^2 g.$$
$= $ quadratic form.

~~Fact~~:
Notation: If we use (u^1, u^2) as coord.
then we let $h_{11} = \mathrm{II}_p(\vec{X}_1, \vec{X}_1)$

$$h_{12} = \mathrm{II}_p(\vec{X}_1, \vec{X}_2) = \mathrm{II}_p(\vec{X}_2, \vec{X}_1) = h_{21}$$

$$h_{22} = \mathrm{II}_p(\vec{X}_2, \vec{X}_2).$$

If $w = a^1 \vec{X}_1 + a^2 \vec{X}_2$, then

$$\mathrm{II}_p(w, w) = (a^1)^2 h_{11} + 2a^1a^2 h_{12} + (a^2)^2 h_{22}$$
$$= \sum_{i,j=1}^{2} h_{ij} a^i a^j.$$

Fact: a, b, c, d are vectors in $\mathbb{R}^3$
then $\langle a \times b, c \times d \rangle$

$$= \begin{vmatrix} \langle a, c \rangle & \langle a, d \rangle \\ \langle b, c \rangle & \langle b, d \rangle \end{vmatrix}$$

Hence $|X_u \times X_v|^2 = \begin{vmatrix} \langle X_u, X_u \rangle & \langle X_u, X_v \rangle \\ \langle X_v, X_u \rangle & \langle X_v, X_v \rangle \end{vmatrix}$

$$= EG - F^2.$$

Prop 3 (Proof). Here $N = \frac{X_u \times X_v}{|X_u \times X_v|}$.

$$e = \mathrm{II}(X_u, X_u) = -\langle N_u, X_u \rangle$$

$$= \langle N, X_{uu} \rangle. \quad (\text{Why?})$$

$$= \frac{\langle X_u \times X_v, X_{uu} \rangle}{\sqrt{EG - F^2}}$$

$$= \frac{\det(X_u, X_v, X_{uu})}{\sqrt{EG - F^2}}.$$

etc.

In (u^1, u^2) coor.,

$$h_{11} = \frac{\det(X_1, X_2, X_{11})}{\sqrt{h_{11} h_{22} - h_{12}^2}}. \quad \sqrt{\det(h)}$$

$$\det h = \det \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}.$$