1. **The Euler-Lagrange equations**

Consider the action:

$$S = \int_a^b \mathcal{L}(t, \phi, \dot{\phi}) dt$$

Here $\phi = (\phi^1, \ldots, \phi^m)$ is a vector valued function of t, $\dot{\phi} = \frac{d}{dt}\phi$.

$\mathcal{L} = \mathcal{L}(t; u^1, \ldots, u^m; z^1, \ldots, z^m)$ is called Lagrangian. We always assume that $\mathcal{L}$ is smooth in t, u, z in the domain under consideration.

Let us take a variation of the action. Namely, let $\eta(t)$ is a smooth function so that $\eta = 0$ near a, b Let

$$S(\epsilon) = \int_a^b \mathcal{L}(t, \phi + \epsilon\eta, \overbrace{(\phi + \epsilon\eta)}^{\cdot})dt$$

Suppose $\mathcal{L}(t, \phi + \epsilon\eta, \overbrace{(\phi + \epsilon\eta)}^{\cdot})$ is smooth for ϵ is small. Then

$$\begin{aligned}
\frac{d}{d\epsilon} S(\epsilon)|_{\epsilon=0} &= \int_a^b \left(\sum_k \eta^k \frac{\partial \mathcal{L}}{\partial \phi^k} + \sum_{k,\mu} \dot{\eta}^k \frac{\partial \mathcal{L}}{\partial \dot{\phi}^k} \right) dt \\
&= \int_a^b \left(\sum_k \eta^k \left(\frac{\partial \mathcal{L}}{\partial \phi^k} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}^k} \right) \right) \right) dx.
\end{aligned}$$

Let

$$E_k =: \frac{\partial \mathcal{L}}{\partial \phi^k} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}^k} \right)$$

for $k = 1, \ldots, m$. These are called Euler-Lagrange expression (E.-L. expression.

The E.-L. equation is the system $E_k = 0$, i.e.

$$\frac{\partial \mathcal{L}}{\partial \phi^k} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}^k} \right) = 0$$

for $k = 1, \ldots, m$.

Lemma 1. *Let $f = (f_1, \ldots, f_m)$ be a vector valued continuous functions on $[a, b]$ such that*

$$\int_a^b \sum_k f_k \eta_k dt = 0$$

for any smooth functions η_k with compact supports in (a, b), i.e. $\eta_k = 0$ near a, b. Then $f_k = 0$ for all k.

Theorem 1. *A C^2 function $\phi = (\phi^1, \ldots, \phi^m)$ which satisfies the E-L equations for $\mathcal{L}$ above if and only if it is an extremal for S. That is $S'(0) = 0$ for all smooth variation.*

Example: Consider m particles in three space with coordinates (x^j, y^j, z^j) with mass $\mathfrak{m}_j$. Then

$$\mathcal{L} = \frac{1}{2}\sum_j \mathfrak{m}_j \left[(\dot{x}^j)^2 + (\dot{y}^j)^2 + (\dot{z}^j)^2\right] - V(t, x, y, z)$$

where V is the potential energy. Here ϕ^k are those x^j, y^j, z^j which depend only on t. $\dot{\phi}^k$ are those $\dot{x}^j$, etc.

E.-L. expressions are given by

$$E_{1j} = -\frac{\partial V}{\partial x^j} - \mathfrak{m}_j \frac{d^2 x^j}{dt^2}; E_{2j} = -\frac{\partial V}{\partial y^j} - \mathfrak{m}_j \frac{d^2 y^j}{dt^2}; E_{3j} = -\frac{\partial V}{\partial z^j} - \mathfrak{m}_j \frac{d^2 z^j}{dt^2}.$$

2. **Geodesics**

Definition 1. Let M be a regular surface. It is said to be a *geodesic* if and only if it is an extremal of the length functional with respect to any smooth variation which vanishes near a, b and is *parametrized proportional to arc length.*

Then length functional is defined as

$$\ell(\alpha) = \int_a^b |\dot{\alpha}| dt$$

if $\alpha : [a, b] \to M$ is a regular curve. α is parametrized proportional to arc length if $|\dot{\alpha}|$ is constant in $[a, b]$.

Suppose $\mathbf{X} : U \to M$ is a coordinate chart, with $(u^1, u^2) \to \mathbf{X}(u^1, u^2)$. We want to find the equations for $u^1(t), u^2(t)$ so that $\alpha(t) = \mathbf{X}(u^1(t), u^2(t))$ is a pregeodisic or geodesic.

Let $g_{ij} = \langle \mathbf{X}_i, \mathbf{X}_j \rangle$ be the first fundamental form. The Lagrangian $\mathcal{L}$ of the length functional is

$$\mathcal{L} = \left(\sum_{i,j=1}^{2} g_{ij} \dot{u}^i \dot{u}^j \right)^{\frac{1}{2}}$$

g_{ij} is a function of u^1, u^2. So $\mathcal{L}$ is a function of u^i, z^i, where z^i corresponding to $\dot{u}^i$. It is smooth as long as (z^1, z^2) is not zero and $(u^1, u^2) \in U$.

Let Γ_{ij}^k be the Christoffel symbols.

Lemma 2. *Let $\mathbf{X} : U \to M$ is a coordinate chart with coordinates (u^1, u^2) Let $\alpha(t) = \mathbf{X}(u^1(t), u^2(t))$ be a regular curve on M. If α is an extremal of the length functional, then u^1, u^2 satisfy the following* *differential equations:*

$$
\begin{cases} \ddot{u}^1 + \sum_{i,j=1}^2 \Gamma_{ij}^1 \dot{u}^i \dot{u}^j = & \lambda \dot{u}^1; \\ \ddot{u}^2 + \sum_{i,j=1}^2 \Gamma_{ij}^2 \dot{u}^i \dot{u}^j = & \lambda \dot{u}^2 \end{cases} \tag{1}
$$

in $[a,b]$, *where*

$$
\lambda = -\frac{1}{2}\left(\frac{1}{2}\frac{d}{dt}\log\left(g_{pq}\dot{u}^p\dot{u}^q\right)\right).
$$

Remark: The equations can also be written as

$$
\ddot{u}^{\mathbf{k}} + \sum_{i,j=1}^{2} \Gamma_{ij}^{\mathbf{k}} \dot{u}^i \dot{u}^j = \lambda \dot{u}^{\mathbf{k}}
$$

for $k = 1, 2$.

Proof. (Sketch) Lagrangian $\mathcal{L} = (g_{ij}\dot{u}^i\dot{u}^j)^{\frac{1}{2}}$ which is smooth in $(u^1, u^2) \in U$ and $(\dot{u}^1, \dot{u}^2) \neq 0$. For any smooth functions η^1, η^2 of t, the curve $\alpha(t, \epsilon) = \mathbf{X}(u^1(t) + \epsilon\eta^1(t), u^2(t) + \epsilon\eta^2(t))$ is regular and so the tangent vectors are nonzero. Hence $\mathcal{L}$ is smooth for this values. Hence if α is an extremal, then $u^1(t), u^2(t)$ should satisfies the E-L equations. On the other hand,

$$
\frac{\partial}{\partial u^k}\mathcal{L} = \frac{1}{2}\left(g_{pq}\dot{u}^p\dot{u}^q\right)^{-\frac{1}{2}} \frac{\partial g_{ij}}{\partial u^k}\dot{u}^i\dot{u}^j.
$$

$$
\frac{\partial}{\partial \dot{u}^k}\mathcal{L} = \left(g_{pq}\dot{u}^p\dot{u}^q\right)^{-\frac{1}{2}} g_{kj}\dot{u}^j.
$$

Hence the E-L equations are:

$$
\frac{1}{2}\left(g_{pq}\dot{u}^p\dot{u}^q\right)^{-\frac{1}{2}} \frac{\partial g_{ij}}{\partial u^k}\dot{u}^i\dot{u}^j - \frac{d}{dt}\left[\left(g_{pq}\dot{u}^p\dot{u}^q\right)^{-\frac{1}{2}} g_{kj}\dot{u}^j\right] = 0.
$$

Now

$$
\frac{d}{dt}\left[\left(g_{pq}\dot{u}^p\dot{u}^q\right)^{-\frac{1}{2}} g_{kj}\dot{u}^j\right] = \left(g_{kj}\dot{u}^j\right)\frac{d}{dt}\left(g_{pq}\dot{u}^p\dot{u}^q\right)^{-\frac{1}{2}} + \left(g_{pq}\dot{u}^p\dot{u}^q\right)^{-\frac{1}{2}}\left(\frac{\partial g_{kj}}{\partial u^r}\dot{u}^r\dot{u}^j + g_{kj}\ddot{u}^j\right).
$$

Hence we have

$$
\left(\frac{\partial g_{kj}}{\partial u^i}\dot{u}^i\dot{u}^j + g_{kj}\ddot{u}^j\right) - \frac{1}{2}\frac{\partial g_{ij}}{\partial u^k}\dot{u}^i\dot{u}^j = -\frac{1}{2}\left(g_{kj}\dot{u}^j\right)\frac{d}{dt}\log\left(g_{pq}\dot{u}^p\dot{u}^q\right)
$$

Multiply both sides by g^{kl} and sum on k, we have

$$
\ddot{u}^l + g^{kl}g_{kj,i}\dot{u}^i\dot{u}^j - \frac{1}{2}g^{kl}g_{ij,k}\dot{u}^i\dot{u}^j = -\dot{u}^l\left(\frac{1}{2}\frac{d}{dt}\log\left(g_{pq}\dot{u}^p\dot{u}^q\right)\right).
$$

From this the result follows. ∎

Theorem 2. *With the above notations, a regular curve α in M is a geodesic if and only if*

$$\begin{cases} \ddot{u}^1 + \sum_{i,j=1}^2 \Gamma_{ij}^1 \dot{u}^i \dot{u}^j = & 0; \\ \ddot{u}^2 + \sum_{i,j=1}^2 \Gamma_{ij}^2 \dot{u}^i \dot{u}^j = & 0. \end{cases} \tag{2}$$

Proof. If α is a geodesic, then it is an extremal for the length functional and is parametrized proportional to arc length. By Lemma 2, we conclude that it satisfies (2).

Conversely, if α satisfies (2), then we want to prove that it is parametrized proportional to arc length. If this is true. Then α is a geodesic by By Lemma 2. **We will prove that α is parametrized proportional to arc length in the next section**. □

Remark: A regular curve $\mathbf{X}(u^1(t), u^2(t))$ is said to be a *pregeodesic* if it satisfies (1) for some continuous function λ.

3. Geodesics and geodesic curvature

Recall the following: Let M be an oriented regular surface with unit normal vector field $\mathbf{N}$. Let $\alpha : [a, b] \to M$ be a regular curve parametrized by arc length. Let $\mathbf{n}$ be the unit normal vector field along α so that $\alpha', \mathbf{n}, \mathbf{N}$ are positively oriented. Then

$$\alpha'' = k_n \mathbf{N} + k_g \mathbf{n}$$

where k_n is called the normal curvature of α and k_g is called the geodesic curvature of α. We want to find k_g.

Now let $\mathbf{X} : U \to M$ be a coordinate chart with coordinates u^1, u^2. Let g_{ij} be the first fundamental form, and Γ_{ij}^k be the Christoffel symbols.

Proposition 1. *Let $\alpha(t)$ be a regular curve in $\mathbf{X}(U)$ so that $\alpha(t) = \mathbf{X}(u^1(t), u^2(t))$. Then*

$$\alpha''(t) = \sum_{k=1}^2 \mathbf{X}_k \left(u_k'' + \sum_{i,j=1}^2 \Gamma_{ij}^k u_i' u_j' \right) + c\mathbf{N}$$

for some function $c(t)$. In particular, if α satisfies (2), then t is proportional to arc length, i.e. $|\alpha'|$ =constant.

Moreover, if α is parametrized by arc length, $\mathbf{N} = \mathbf{X}_1 \times \mathbf{X}_1 / |\mathbf{X}_1 \times \mathbf{X}_2|$, then

$$\alpha''(t) = \sum_{k=1}^2 \mathbf{X}_k \left(u_k'' + \sum_{i,j=1}^2 \Gamma_{ij}^k u_i' u_j' \right) + k_n \mathbf{N}.$$

Hence if α is parametrized proportional to arc length, then α is a geodesic if and only if its geodesic curvature is zero.

Proof. (Sketch) $\alpha(t) = \mathbf{X}(u_1(t), u_2(t))$ is a curve on M. Then $\alpha' = u_1'\mathbf{X}_1 + u_2'\mathbf{X}_2$.

$$\begin{aligned}
\alpha'' =& u_1''\mathbf{X}_1 + u_2''\mathbf{X}_2 + (u_1')^2\mathbf{X}_{11} + 2u_1'u_2'\mathbf{X}_{12} + (u_2')^2\mathbf{X}_{22}) \\
=& \mathbf{X}_1\left(u_1'' + \Gamma_{11}^1(u_1')^2 + 2\Gamma_{12}^1 u_1'u_2' + \Gamma_{22}^1(u_2')^2\right) \\
&+ \mathbf{X}_2\left(u_2'' + \Gamma_{11}^2(u_1')^2 + 2\Gamma_{12}^2 u_1'u_2' + \Gamma_{22}^2(u_2')^2\right) \\
&+ c\mathbf{N} \\
=& \sum_{k=1}^2 \mathbf{X}_k \left(u_k'' + \sum_{i,j=1}^2 \Gamma_{ij}^k u_i'u_j'\right) + c\mathbf{N}.
\end{aligned}$$

□

Examples: Let M be a regular surface in $\mathbb{R}^3$. Let α be a regular curve with $|\alpha'| = 1$ in M such that $\alpha = P \cap M$ where P is some hyperplane so that $P \perp M$. Then α is a geodesic.

- Great circles of spheres are geodesics.
- Meridians of a surface revolution are geodesics. *Questions: How about parallels?*

Before we state the next fact, we need to define an oriented surface without referring to $\mathbf{N}$.

Definition 2. A regular surface M is said to be orientable if it can be covered by coordinate charts so that the coordinate transformation (or reparametrization) is orientation preserving.

Proposition 2. *Geodesic curvature is intrinsic in the sense that it depends only on the first fundamental form and the orientation of a regular surface.*

Assignment 7, Due Friday Nov 9, 2018

(1) Suppose a regular surface M is parametrized by u^1, u^2 so that the first fundamental form is given by

$$g_{11} = g_{22} = \frac{1}{1 - \frac{1}{4}\sum_{i=1}^2 (u^i)^2}, \quad g_{12} = 0.$$

Find the Gaussian curvature of the surface. Here we assume that $(u^1)^2 + (u^2)^2 < 4$.

(2) Find the geodesics on the circular cylinder:$M = \{(x, y, z)| \; x^2 + y^2 = r^2\}$ where $r > 0$ is a constant. Here you may use the parametrization of M as

$$\mathbf{X}(u^1, u^2) = (\cos u^1, \sin u^1, u^2).$$

A regular curve can be expressed as $\alpha(t) = (r\cos u^1(t), r\sin u^1(t), u^2(t))$,

(3) Suppose α is a pregeodsic. Namely, α satisfies (1) in the note for some continuous function λ. Let $f(t)$ be functions so that $f' = -\lambda$ and $F' = e^f$. Namely

$$F = \int \exp(\int \lambda).$$

Reparametrized a by τ so that $\tau = F(t)$.

Prove that $\alpha(t) = \alpha(t(\tau))$ as a curve parametrized by τ is a geodesic.

(4) Suppose a regular surface M is parametrized by u^1, u^2 so that the first fundamental form is given by

$$g_{11} = g_{22} = \frac{1}{\left(1 - \frac{1}{4}\sum_{i=1}^{2}(u^i)^2\right)^2}, g_{12} = 0.$$

Find the Gaussian curvature of the surface. Here we assume that $(u^1)^2 + (u^2)^2 < 4$.