1. **Gauss map and Gaussian curvature**

Definition 1. Let M be a regular surface. M is said to be *orientable* if there is a continuous (and hence smooth) unit normal vector field $\mathbf{N}$ defined on M and M is said to be oriented by $\mathbf{N}$.

Let M be an orientable regular surface and let $\mathbf{N}$ be a unit normal vector field. The Gauss map is defined as a map from M to the unit sphere in $\mathbb{R}^3$ by:

$$\mathbf{N} : M \to \mathbb{S}^2.$$

Let $\mathbf{X} : U \to M$ be a regular surface patch and et Q be a bounded region in U. Let $R = \mathbf{X}(Q)$. Then the area of R is given by:

$$A(R) = \int_Q |\mathbf{X}_u \wedge \mathbf{X}_v| dudv.$$

Proposition 1. *Let $p \in M$. Suppose $K(p) \neq 0$. Let B_n be a sequence of open sets with $B_n \to p$ in the sense that $\sup_{q \in B_n} |p - q| \to 0$ as $n \to \infty$. Let A_n be the area of B_n and $\widetilde{A}_n$ be the area of the Gauss image $\mathbf{n}(B_n)$ of B_n. Then*

$$\lim_{n\to\infty} \frac{\widetilde{A}_n}{A_n} = |K(p)|.$$

Proof. (sketch) Let $\mathbf{X}(u, v)$ be a coordinate paramatrization near p such that $\mathbf{X}(0, 0) = p$. Let

$$\mathcal{S}_p(\mathbf{X}_u) = a_1^1\mathbf{X}_u + a_1^2\mathbf{X}_v, \ \mathcal{S}_p(\mathbf{X}_v) = a_2^1\mathbf{X}_u + a_2^2\mathbf{X}_v,$$

then $\det(a_i^j) = K$. Suppose U_n in the (u, v) plane such that $\mathbf{X}(U_n) = B_n$. Then

$$A_n = \iint_{U_n} |\mathbf{X}_u \times \mathbf{X}_v| dudv,$$

and

$$\widetilde{A}_n = \iint_{U_n} |\mathbf{N}_u \times \mathbf{N}_v| dudv.$$

where $\mathbf{N} = \mathbf{X}_u \times \mathbf{X}_v / |\mathbf{X}_u \times \mathbf{X}_v|$ is the unit normal on M. Now

$$\mathbf{N}_u = -\mathcal{S}_p(\mathbf{X}_u), \mathbf{N}_v = -\mathcal{S}_p(\mathbf{X}_v).$$

Hence

$$\mathbf{N}_u \times \mathbf{N}_v = \det(a_i^j)\mathbf{X}_u \times \mathbf{X}_v = K\mathbf{X}_u \times \mathbf{X}_v.$$

So

$$\begin{aligned}|\mathbf{N}_u \times \mathbf{N}_v|(u, v) =& |K(u, v)||\mathbf{X}_u \times \mathbf{X}_v|(u, v) \\ =& |K(0, 0)||\mathbf{X}_u \times \mathbf{X}_v|(u, v) + (|K(u, v) - |K(0, 0)|) |\mathbf{X}_u \times \mathbf{X}_v|(u, v).\end{aligned}$$

Since $B_n \to p$, we have $U_n \to (0,0)$. Hence for any $\epsilon > 0$ there is $N > 0$ such that if $n \geq N$, then $||K(u,v) - |K(0,0)|| \leq \epsilon$.

$$\widetilde{A}_n = |K(0,0)|A_n + R_n$$

where $|R_n| \leq \epsilon A_n$, if $n \geq N$. From this it is easy to see the proposition follows. □

2. Theorema Egregium of Gauss

Definition 2. Let $F : M_1 \to M_2$ be a diffeomorphism. F is said to be an *isometry* if for any $p \in M_1$ and $q = F(p)$, the linear map $dF : M_1 \to M_2$ is an isometry as inner product spaces. If there is an isometry from M_1 onto M_2, then M_1 is said to be isometric to M_2.

Theorem 1. *(Theorema Egregium of Gauss) The Guassian curvature K is invariant under isometries. That is to say, the Gaussian curvature depends only on the first fundamental form.*

Proof. (First proof.) Let $\mathbf{X}(u,v)$ be a local parametrization of a regular surface, and let E, F, G be the coefficients of the first fundamental form and e, f, g be the second fundamental form. In the following, if a, b, c are three vectors, (a,b,c) is the ordered triple product of the three vectors. Now

$$e = \langle \mathbf{N}, \mathbf{X}_{uu} \rangle = \frac{(\mathbf{X}_{uu}, \mathbf{X}_u, \mathbf{X}_v)}{\sqrt{EG - F^2}},$$

etc.

$$\begin{aligned} K &= \frac{eg - f^2}{EG - F^2} \\ &= \frac{[(\mathbf{X}_{uu}, \mathbf{X}_u, \mathbf{X}_v)(\mathbf{X}_{vv}, \mathbf{X}_u, \mathbf{X}_v) - (\mathbf{X}_{uv}, \mathbf{X}_u, \mathbf{X}_v)^2]}{(EG - F^2)^2}. \end{aligned}$$

Hence

$(EG - F^2)^2 K$

$= \det(\mathbf{X}_{uu}, \mathbf{X}_u, \mathbf{X}_v) \det(\mathbf{X}_{vv}, \mathbf{X}_u, \mathbf{X}_v) - (\det(\mathbf{X}_{uv}, \mathbf{X}_u, \mathbf{X}_v))^2$

$$= \begin{vmatrix} \langle \mathbf{X}_{uu}, \mathbf{X}_{vv} \rangle & \langle \mathbf{X}_{uu}, \mathbf{X}_u \rangle & \langle \mathbf{X}_{uu}, \mathbf{X}_v \rangle \\ \langle \mathbf{X}_u, \mathbf{X}_{vv} \rangle & \langle \mathbf{X}_u, \mathbf{X}_u \rangle & \langle \mathbf{X}_u, \mathbf{X}_v \rangle \\ \langle \mathbf{X}_v, \mathbf{X}_{vv} \rangle & \langle \mathbf{X}_v, \mathbf{X}_u \rangle & \langle \mathbf{X}_v, \mathbf{X}_v \rangle \end{vmatrix} - \begin{vmatrix} \langle \mathbf{X}_{uv}, \mathbf{X}_{uv} \rangle & \langle \mathbf{X}_{uv}, \mathbf{X}_u \rangle & \langle \mathbf{X}_{uv}, \mathbf{X}_v \rangle \\ \langle \mathbf{X}_u, \mathbf{X}_{uv} \rangle & \langle \mathbf{X}_u, \mathbf{X}_u \rangle & \langle \mathbf{X}_u, \mathbf{X}_v \rangle \\ \langle \mathbf{X}_v, \mathbf{X}_{uv} \rangle & \langle \mathbf{X}_v, \mathbf{X}_u \rangle & \langle \mathbf{X}_v, \mathbf{X}_v \rangle \end{vmatrix}$$

$$= \begin{vmatrix} \langle \mathbf{X}_{uu}, \mathbf{X}_{vv} \rangle - \langle \mathbf{X}_{uv}, \mathbf{X}_{uv} \rangle & \langle \mathbf{X}_{uu}, \mathbf{X}_u \rangle & \langle \mathbf{X}_{uu}, \mathbf{X}_v \rangle \\ \langle \mathbf{X}_u, \mathbf{X}_{vv} \rangle & \langle \mathbf{X}_u, \mathbf{X}_u \rangle & \langle \mathbf{X}_u, \mathbf{X}_v \rangle \\ \langle \mathbf{X}_v, \mathbf{X}_{vv} \rangle & \langle \mathbf{X}_v, \mathbf{X}_u \rangle & \langle \mathbf{X}_v, \mathbf{X}_v \rangle \end{vmatrix}$$

$$- \begin{vmatrix} 0 & \langle \mathbf{X}_{uv}, \mathbf{X}_u \rangle & \langle \mathbf{X}_{uv}, \mathbf{X}_v \rangle \\ \langle \mathbf{X}_u, \mathbf{X}_{uv} \rangle & \langle \mathbf{X}_u, \mathbf{X}_u \rangle & \langle \mathbf{X}_u, \mathbf{X}_v \rangle \\ \langle \mathbf{X}_v, \mathbf{X}_{uv} \rangle & \langle \mathbf{X}_v, \mathbf{X}_u \rangle & \langle \mathbf{X}_v, \mathbf{X}_v \rangle \end{vmatrix}$$

Now

$$\langle \mathbf{X}_{uu}, \mathbf{X}_v \rangle = \langle \mathbf{X}_u, \mathbf{X}_v \rangle_u - \langle \mathbf{X}_u, \mathbf{X}_{vu} \rangle = F_u - \frac{1}{2} E_v.$$

$$\langle \mathbf{X}_u, \mathbf{X}_{vv} \rangle = \langle \mathbf{X}_u, \mathbf{X}_v \rangle_v - \langle \mathbf{X}_{uv}, \mathbf{X}_v \rangle = F_v - \frac{1}{2} G_u.$$

$$\begin{aligned} &\langle \mathbf{X}_{uu}, \mathbf{X}_{vv} \rangle - \langle \mathbf{X}_{uv}, \mathbf{X}_{uv} \rangle \\ &\quad = \langle \mathbf{X}_{uu}, \mathbf{X}_v \rangle_v - \underline{\langle \mathbf{X}_{uuv}, \mathbf{X}_v \rangle} - \langle \mathbf{X}_{uv}, \mathbf{X}_v \rangle_u + \underline{\langle \mathbf{X}_{uvu}, \mathbf{X}_v \rangle} \\ &\quad = F_{uv} - \frac{1}{2} E_{vv} - \frac{1}{2} G_{uu} \end{aligned}$$

Now

$$\begin{aligned} \langle \mathbf{X}_{uu}, \mathbf{X}_{vv} \rangle - \langle \mathbf{X}_{uv}, \mathbf{X}_{uv} \rangle &= \langle \mathbf{X}_{uu}, \mathbf{X}_v \rangle_v - \langle \mathbf{X}_{uuv}, \mathbf{X}_v \rangle - \langle \mathbf{X}_{uv}, \mathbf{X}_v \rangle_u + \langle \mathbf{X}_{uvu}, \mathbf{X}_v \rangle \\ &= \langle \mathbf{X}_{uu}, \mathbf{X}_v \rangle_v - \frac{1}{2} G_{uu} \\ &= \langle \mathbf{X}_u, \mathbf{X}_v \rangle_{uv} - \langle \mathbf{X}_u, \mathbf{X}_{vu} \rangle_v - \frac{1}{2} G_{uu} \\ &= F_{uv} - \frac{1}{2} E_{vv} - \frac{1}{2} G_{uu}. \end{aligned}$$

Hence we have

$$K = \frac{A - B}{(EG - F^2)^2}$$

where

$$A = \begin{vmatrix} -\frac{1}{2} E_{vv} + F_{uv} - \frac{1}{2} G_{uu} & \frac{1}{2} E_u & F_u - \frac{1}{2} E_v \\ F_v - \frac{1}{2} G_u & E & F \\ \frac{1}{2} G_v & F & G \end{vmatrix},$$

and

$$B = \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}.$$

□

3. **Christoffel symbols**

Let $\mathbf{X}(u^1, u^2)$ is a coordinate parametrization. Let $\mathbf{X}_i = \mathbf{X}_{u^i}$, $g_{ij} = \langle \mathbf{X}_i, \mathbf{X}_j \rangle$, $(g^{ij}) = (g_{ij})^{-1}$. Then

$$\mathbf{X}_{ij} = h_{ij}\mathbf{N} + \Gamma_{ij}^k \mathbf{X}_k \tag{1}$$

(Einstein summation convention: repeated indices mean summation.)

Γ_{ij}^k are called the *Christoffel symbols* for this parametrization.

Lemma 1. $\Gamma_{ij}^k = \Gamma_{ji}^k$ *and*

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^{2} g^{kl}\left(g_{il,j} + g_{jl,i} - g_{ij,l}\right).$$

where $g_{ij,l} = \frac{\partial}{\partial u^l} g_{ij}$ *etc.*

Proof. (Sketch)

$$\langle \mathbf{X}_{ij}, \mathbf{X}_l \rangle = \Gamma_{ij}^k g_{kl}$$

So

$$g_{il,j} - \langle \mathbf{X}_i, \mathbf{X}_{lj} \rangle = \Gamma_{ij}^k g_{kl}$$

So

$$g_{il,j} = \Gamma_{ij}^k g_{kl} + \Gamma_{lj}^k g_{ki}.$$

Hence we have

$$\begin{cases} g_{il,j} = & \Gamma_{ij}^k g_{kl} + \underline{\Gamma_{lj}^k g_{ki}}. \\ g_{jl,i} = & \Gamma_{ji}^k g_{kl} + \underline{\underline{\Gamma_{li}^k g_{kj}}}. \\ g_{ij,l} = & \underline{\underline{\Gamma_{il}^k g_{kj}}} + \underline{\Gamma_{jl}^k g_{ki}}. \end{cases}$$

Hence

$$g_{il,j} + g_{jl,i} - g_{ij,l} = 2\Gamma_{ij}^k g_{kl}.$$

From this the result follows.

□

4. **Formula for Gaussian curvature**

We give another proof of the Theorema Egregium of Gauss.

Theorem 2. *With the above notations, then*

$$2K = g^{ij}\left(\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^k_{lk}\Gamma^l_{ji} - \Gamma^k_{lj}\Gamma^l_{ki}\right) = g^{ij}(\Gamma^k_{i[j,k]} + \Gamma^k_{l[k}\Gamma^l_{j]i}).$$

Here $T_{[ij]k} = T_{ijk} - T_{jik}$ *etc.*

Proof. (Sketch)

Let $\mathcal{S}$ be the shape operator, then

$$-\mathbf{N}_i = \mathcal{S}(\mathbf{X}_i) = a^j_i \mathbf{X}_j.$$

$$\begin{aligned}\mathbf{X}_{ijm} =& h_{ij,m}\mathbf{N} + h_{ij}\mathbf{N}_m + \Gamma^k_{ij,m}\mathbf{X}_k + \Gamma^k_{ij}\mathbf{X}_{km} \\ =& \left(h_{ij,m} + \Gamma^k_{ij}h_{km}\right)\mathbf{N} + \left(-h_{ij}a^k_m + \Gamma^k_{ij,m} + \Gamma^s_{ij}\Gamma^k_{sm}\right)\mathbf{X}_k\end{aligned}$$

Since $\mathbf{X}_{ijm} = \mathbf{X}_{imj}$, we have

$$\left(-h_{ij}a^k_m + \Gamma^k_{ij,m} + \Gamma^s_{ij}\Gamma^k_{sm}\right)\mathbf{X}_k = \left(-h_{im}a^k_j + \Gamma^k_{im,j} + \Gamma^s_{im}\Gamma^k_{sj}\right)\mathbf{X}_k$$

So

$$\Gamma^k_{ij,m} + \Gamma^s_{ij}\Gamma^k_{sm} - \Gamma^k_{im,j} - \Gamma^s_{im}\Gamma^k_{sj} = -h_{im}h_{kj} + h_{ij}h_{km}$$

Or

$$h_{ij}a^k_m - h_{im}a^k_j = \Gamma^k_{i[j,m]} + \Gamma^s_{i[j}\Gamma^k_{m]s}$$

Let $m = k$, we have

$$\sum_{i,j,k} g^{ij}\left(h_{ij}a^k_k - h_{ik}a^k_j\right) = g^{ij}\left(\Gamma^k_{i[j,k]} + \Gamma^s_{i[j}\Gamma^k_{k]s}\right).$$

Now the matrix of the shape operator is:

$$(a^j_i) = (h_{ij})(g_{ij})^{-1}$$

So $h_{ji} = h_{ij} = a^l_i g_{lj}$. Hence

$$\begin{aligned}\sum_{i,j,k} g^{ij}\left(h_{ij}a^k_k - h_{ik}a^k_j\right) =& H^2 - g^{ij}a^l_k g_{li}a^k_j \\ =& H^2 - \sum_{j,k} a^k_j a^j_k \\ =& (a^1_1 + a^2_2)^2 - \left((a^1_1)^2 + (a^2_2)^2 + 2a^2_1 a^1_2\right) \\ =& 2\left(a^1_1 a^2_2 - a^2_1 a^1_2\right) \\ =& 2K.\end{aligned}$$

□

Assignment 6, Due Friday Nov 2, 2018

(1) Prove that if $\mathbf{X}$ is an orthogonal parametrization, i.e. $F = 0$, then the Gaussian curvature is given by:

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right].$$

Suppose in addition $E = G$ everywhere, then

$$K = -e^{-2f}\Delta f$$

where f is such that $E = e^{2f}$ (i.e. $f = \frac{1}{2}\log E$), and Δ is the Laplacian operator:

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}.$$

(2) Compute the Christoffel symbols for a surface of revolution:

$$\mathbf{X}(u^1, u^2) = (f(u^2)\cos u^1, f(u^2)\sin u^1, g(u^2))$$

with $f > 0$.

(3) Verify that the surfaces:

$$\mathbf{X}(u, v) = (u\cos v, u\sin v, \log u)$$

and

$$\mathbf{Y}(u, v) = (u\cos v, u\sin v, v)$$

have equal Gaussian curvature at that points $\mathbf{X}(u, v), \mathbf{Y}(u, v)$ but the coefficients of the first fundamental forms at points $\mathbf{X}(u, v), \mathbf{Y}(u, v)$ are not the same.