Reminder: Mid-term exam will be on Oct 16, Tuesday, 8:30am–10:15am.

Regular surfaces

Definition 1. A subset $M \subset \mathbb{R}^3$ is said to be a *regular surface* if for any $p \in M$, there is an open neighborhood U of p in M, an open set D in $\mathbb{R}^2$ and a map $\mathbf{X} : D \to M \cap U$ such that the following are true:

(rs1) $\mathbf{X}$ is smooth.

(rs2) $\mathbf{X}$ is full rank: $\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u}$ and $\mathbf{X}_v = \frac{\partial \mathbf{x}}{\partial v}$ are linearly independent, for any $(u, v) \in D$.

(rs3) $\mathbf{X}$ is a homeomorphism from D onto $M \cap U$. (That is: $\mathbf{X}$ is bijective, $\mathbf{X}$ and $\mathbf{X}^{-1}$ are continuous).

Let M be a regular surface, a map $\mathbf{X} : U \to V$ where V is an open set of M, satisfying the above conditions. $\mathbf{X}$ is called a *parametrization (a system of local coordinates)*, and V is called a *coordinate chart (patch)*. If $\mathbf{X}(u, v) = p$, then (u, v) are called local coordinates of p.

1. Basic properties and definitions

Proposition 1. *Let M be regular surface and let $\mathbf{X} : U \to M$ be a coordinate parametrization. Then for any $p = (u_0, v_0) \in U$ there is a open set $V \subset U$ with $p \in V$ such that $\mathbf{X}(V)$ is a graph over an open set in one of the coordinate plane.*

Proposition 2. *(Change of coordinates) Let M be a regular surface and let $\mathbf{X} : U \to M$, $\mathbf{Y} : V \to M$ be two coordinate parametrizations. Let $S = \mathbf{X}(U) \cap \mathbf{Y}(V) \subset M$ and let $U_1 = \mathbf{X}^{-1}(S)$ and $V_1 = \mathbf{Y}^{-1}(S)$. Then $\mathbf{Y}^{-1} \circ \mathbf{X} : U_1 \to V_1$ is a diffeomorphism.*

Proof. (Sketch) Let $p \in S$. Then there is an open set $S_1 \subset S$ such that S_1 is given by the graph $\{(x, y, z) | (x, y) \in \mathcal{O}, z = f(x, y)\}$. Now if $(u, v) \in U_1$ with $\mathbf{X}(u, v) \in S_1$, then

$$\mathbf{X}(u, v) = (x(u, v), y(u, v), f(x(u, v), y(u, v))$$

because $z = f(x, y)$. Then

$$\mathbf{X}_u = (x_u, y_u, f_x x_u + f_y y_u), \mathbf{X}_v = (x_v, y_v, f_x x_v + f_y y_v).$$

Since $\mathbf{X}_u$ and $\mathbf{X}_v$ are linearly independent, we have $(x_u, y_u), (x_v, y_v)$ are linearly independent (why?). This implies $(u, v) \to (x, y)$ is difffeormphic near $\mathbf{X}^{-1}(p)$. Similarly, if $(\xi, \eta) \in V_1$, then $(\xi, \eta) \to (x, y)$ is diffeomorphic near $\mathbf{Y}^{-1}(p)$. Hence $(\xi, \eta) \to (u, v)$ is diffeomorphic. □

Proposition 3. *Let U be an open set in $\mathbb{R}^3$ and let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Suppose a is a regular value of f. (That is: if $f(x) = a$, then $\nabla f(x) \neq \mathbf{0}$.) Then*

$$M = \{x \in U |\ f(x) = a\}$$

is a regular surface.

Definition 2.

(i) Let M be regular surface and let $f : M \to \mathbb{R}$ be a function. f is said to be smooth if and only if $f \circ \mathbf{X}$ is smooth for all coordinate chart $\mathbf{X} : U \to M$.

(ii) M_1, M_2 be regular surfaces and let $F : M_1 \to M_2$ be a map. F is said to be smooth if and only if the following is true: For any $p \in M_1$ and any coordinate charts $\mathbf{X}$ of p, $\mathbf{Y}$ of $q = F(p)$, $\mathbf{Y}^{-1} \circ \mathbf{X}$ is smooth whenever it is defined.

(iii) Suppose $F : M_1 \to M_2$ is a smooth map. Then the differential dF is defined as follows: Suppose $p \in M_1$ and $q = F(p)$. Then $dF_p : T_p(M_1) \to T_q(M_2)$ so that if $\mathbf{v} \in T_p(M_1)$ and α is a smooth curve on M_1 with $\alpha(0) = p$, $\alpha'(0) = \mathbf{v}$, then

$$dF_p(\mathbf{v}) = \frac{d}{dt} F \circ \alpha(t)|_{t=0}.$$

Proposition 4. *The above definitions are well-defined.*

Assignment 5, Due Friday, 26/10/2018

(1) Show that the by the Gaussian curvature K and mean curvature H satisfies $H^2 - K \geq 0$. Show also that the principal curvatures are given by

$$k_1 = H + \sqrt{H^2 - K}, \quad k_2 = H - \sqrt{H^2 - K}.$$

(2) Prove that the mean curvature at a point is given by:

$$H = \frac{1}{2\pi} \int_0^{2\pi} k_n(\theta) d\theta$$

where $k_n(\theta)$ is the normal curvature along a direction making an angle θ with a fixed direction.

(3) An asymptotic direction at p in a regular surface patch M is a direction of $T_p(M)$ for which the normal curvature is zero. Show that at a hyperbolic point, the principal directions bisect the asymptotic direction.

(4) Let M be a regular surface patch. Suppose M is inside a sphere $\mathbb{S}(r)$ of radius $r > 0$ such that M is tangent to $\mathbb{S}(r)$ at a point p. Show that the Gaussian curvature of M at p is at least $1/r^2$.