More on principal curvatures and normal curvatures

Proposition 3. *(Euler formula) Let* e_1, e_2 *be orthonormal in* $T_p(M)$ *where* M *is a regular surface patch. Suppose* e_1, e_2 *are in principal direction with principal curvatures* k_1, k_2 *respectivel. Let* $v = e_1 \cos\theta + e^2 \sin\theta$ *be a unit vector in* $T_p(M)$*. Then the normal curvature in the direction* v *is given by* $k_1 \cos^2\theta + k_2 \sin^2\theta$*.*

Proposition 4. *Let* $\mathbf{X} : U \to \mathbb{R}^3$ *be a regular surface patch and let* $M = \mathbf{X}(U)$*. Suppose every point in* M *is umbilical. Then* M *is contained in a plane or in a sphere.*

Proof. (Sketch) Let $\mathbf{N}$ be a unit normal vector field on M and let $\mathcal{S}$ be the shape operator. Then $\mathcal{S}_p(\mathbf{v}) = \lambda\mathbf{v}$ for any $\mathbf{v} \in T_p(M)$ for some number λ. Here λ may depend on the point p but not $\mathbf{v}$. So we write $\lambda = \lambda(u, v)$.

- λ is constant. That is $\lambda_u = \lambda_v = 0$ everywhere (Why?)
- If $\lambda = 0$, then $\mathbf{N}$ is constant and $\langle \mathbf{X}(u, v) - \mathbf{X}(u_0, v_0), \mathbf{N}\rangle = 0$ for some $(u_0, v_0) \in U$. (Why?)
- If $\lambda \neq 0$, then $\mathbf{X} - \frac{1}{\lambda}\mathbf{N}$ is a constant vector $\mathbf{a}$, say. Then $|\mathbf{X} - \mathbf{a}| = 1/|\lambda|$. (Why?)

□

Definition 3. Let p be a point in a regular surface patch. Then it is called

1. Elliptic if $\det(\mathcal{S}_p) > 0$.
2. Hyperbolic if $\det(\mathcal{S}_p) < 0$
3. Parabolic if $\det(\mathcal{S}_p) = 0$ but $\mathcal{S}_p \neq 0$.
4. Planar if $\mathcal{S}_p = 0$.

Suppose p is not a planar point. Let e_1, e_2 be the principal directions with principal curvature k_1, k_2 with $\mathbf{N} = e_1 \times e_2$. We choose the coordinates in $\mathbb{R}^3$ as follows: p is the origin, e_1 is the positive x-axis direction, e_2 is the positive y-axis direction. Then $\mathbf{X}(U)$ is a graph over xy-plane locally. That is the surface can be expressed as $(x, y, f(x, y))$.

Proposition 5. *Near* $p = (0, 0, 0)$*, the surface the graph of*

$$f(x, y) = \frac{1}{2}(k_1 x^2 + k_2 y^2) + o(x^2 + y^2).$$

Hence locally, the regular surface patch is a

- *elliptic paraboloid if* p *is elliptic;*
- *hyperbolic paraboloid if* p *is hyperbolic;*
- *parabolic cylinder if* p *is parabolic.*

Proof. Since $f(0,0)=0,\ f_x(0,0)=0, f_y(0,0)=0$, we have

$$f(x,y)=\frac{1}{2}(f_{xx}(0,0)x^2+2f_{xy}(0,0)xy+f_{yy}(0,0)y^2)+o(x^2+y^2).$$

Note that $\mathbf{X}_x=(1,0,f_x),\mathbf{X}_y=(0,1,f_y),\mathbf{X}_{xx}=(0,0,f_{xx}),\mathbf{X}_{xy}=\mathbf{X}_{xx}=(0,0,f_{xy}),\mathbf{X}_{yy}=(0,0,f_{yy})$. At p, $f_{xx}=k_1,f_{xy}=0,f_{yy}=k_2$. (Why?)

$\square$