Curves on surfaces

1. Basic facts on symmetric bilinear form

Let $(V, \langle \cdot\ ,\ \cdot \rangle)$ be a finite dimensional inner product space and let B be a *symmetric* bilinear form on V.

- Let Q be the corresponding quadratic form, $Q(\mathbf{v}) = B(\mathbf{v}, \mathbf{v})$
- A be the corresponding self-adjoint operator: $\langle A(\mathbf{v}), \mathbf{w}\rangle = B(\mathbf{v}, \mathbf{w})$.

Theorem 1. *Let $(V, \langle \cdot\ ,\ \cdot \rangle)$ be a finite dimensional inner product space of dimension n and let B be a symmetric bilinear form. Then there is an orthonormal basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ such that B is diagonalized. Namely, $B(\mathbf{v}_i, \mathbf{v}_j) = \lambda_i \delta_{ij}$. $\mathbf{v}_i$ is an eigenvector of A with eigenvalue λ_i: $A(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$. Moreover, if $\mathbf{v} = \sum_{i=1}^n x^i \mathbf{v}_i$, then $Q(\mathbf{v}) = \sum_{i=1}^n \lambda_i (x^i)^2$.*

2. Principal curvatures

Let $\mathbf{X} : U \to \mathbb{R}^3$ be a regular surface patch, and $M = \mathbf{X}(U)$. Let $\mathbf{N}$ be a unit normal vector field on M. Let $p \in T_p(M)$ and let $\mathbb{II}_p$ be the second fundamental form of p at M with respect to $\mathbf{N}$.

Definition 1. Let $\mathbf{e}_1$, $\mathbf{e}_2$ be an orthonormal basis on $T_p(M)$ which diagonalizes $\mathbb{II}_p$ with eigenvalues k_1 and k_2. Then k_1, k_2 are called the *principal curvatures* of M at p and $\mathbf{e}_1$, $\mathbf{e}_2$ are called the principal directions.

Proposition 1. *With the above notations, if $k_1 = k_2 = k$, then every direction is a principal direction and in this case, $\mathcal{S}_p = k\mathbf{id}$. (In this case, the point is said to be umbilical.) Moreover, the Gaussian curvature and the mean curvature are given by $K(p) = k_1 k_2$, and $H(p) = \frac{1}{2}(k_1 + k_2)$.*

3. Normal curvatures

Let M be a regular surface patch. Let $\mathbf{N}$ be a smooth unit normal vector field on M. (*Note: There are two choices of unit normal vector fields).* Let $\alpha(s)$ be a smooth curve on M parametrized by arc length. Let $T = \alpha'$ and let $\mathbf{n}(s)$ be the unit vector at $\alpha(s)$ such that $\mathbf{n} \in T_{\alpha(s)}(M)$ and such that $\{T, \mathbf{n}, \mathbf{N}\}$ is positively oriented, i.e. $\mathbf{n} = \mathbf{N} \times T$.

Lemma 1. *T' is a linear combination of $\mathbf{n}$ and $\mathbf{N}$: $T' = k_g \mathbf{n} + k_n \mathbf{N}$ for some smooth functions k_n and k_g on $\alpha(s)$.*

Definition 2. As in the lemma, $k_n(s)$ is called the *normal curvature* of α at $\alpha(s)$ and $k_g(s)$ is called the *geodesic curvature* of α at $\alpha(s)$.

Facts:

(i) k_n and k_g depend on the choice of $\mathbf{n}$.
(ii) We will see later that k_g is intrinsic: it depends only on the first fundamental form *and* the orientation of the surface.
(iii) Let k be the curvature of α'. Suppose k is not zero. Let N be the normal of α (recalled $\alpha'' = kN$). Then $k_n = k\langle N, \mathbf{N}\rangle = k\cos\theta$ where θ is the angle between N and $\mathbf{N}$. If $k = 0$, then $T' = 0$ and $k_n = k_g = 0$.

For the time being we only discuss normal curvature. The geometric meaning of the second fundamental form is the following:

Proposition 2. *Let M be a regular surface patch and $\mathbf{N}$ be a smooth unit normal vector field on M. Let $\mathrm{I\!I}$ be the second fundamental form of M (w.r.t. $\mathbf{N}$) and let $p \in M$. Suppose $\mathbf{v} \in T_p(M)$ with unit length and suppose $\alpha(s)$ is a smooth curve of M parametrized by arclength with $\alpha(0) = p$ and $\alpha'(0) = \mathbf{v}$. Then*

$$k_n(0) = \mathrm{I\!I}_p(\mathbf{v}, \mathbf{v})$$

where k_n is the normal curvature of α at $\alpha(0) = p$.

Corollary 1. *With the same notation as in the proposition, we have the following:*

(i) *Let α and β be two regular curves parametrized by arc length passing through p. Suppose α and β are tangent at p. Then the normal curvatures of α and β at p are equal.*
(ii) *Let k_1 and k_2 be the eigenvalues of $\mathrm{I\!I}_p$ with $k_1 \le k_2$. Then all normal curvatures are between k_1 and k_2.*

Proof. (On symmetric bilinear form) We just prove the case that $n = 2$. Let S be the set in V with $||\mathbf{v}||^2 = \langle \mathbf{v}, \mathbf{v} \rangle = 1$. Then $B(\mathbf{v}, \mathbf{v})$ attains maximum on S at some $\mathbf{v}$. Let $\mathbf{v}_1 \in S$ be such that

$$B(\mathbf{v}_1, \mathbf{v}_1) = \max_{\mathbf{v} \in S} B(\mathbf{v}, \mathbf{v}).$$

Let $\mathbf{v}_2 \in S$ such that $\mathbf{v}_1 \perp \mathbf{v}_2$. It is sufficient to prove that $B(\mathbf{v}_1, \mathbf{v}_2) = 0$. Let $t \in \mathbb{R}$ and let

$$f(t) = \frac{B(\mathbf{v}_1 + t\mathbf{v}_2, \mathbf{v}_1 + t\mathbf{v}_2)}{||\mathbf{v}_1 + t\mathbf{v}_2||^2}.$$

Then $f'(0) = 0$. Hence

$$\begin{aligned} 0 =& 2B(\mathbf{v}_1, \mathbf{v}_2) - 2B(\mathbf{v}_1, \mathbf{v}_1)\langle \mathbf{v}_1, \mathbf{v}_2 \rangle \\ =& 2B(\mathbf{v}_1, \mathbf{v}_2). \end{aligned}$$

Note that $\lambda_2 = B(\mathbf{v}_2, \mathbf{v}_2) = \min_{\mathbf{v} \in S} B(\mathbf{v}, \mathbf{v})$.

Now $\langle A(\mathbf{v}_1), \mathbf{v}_1 \rangle = B(\mathbf{v}_1, \mathbf{v}_1) = \lambda_1 = \lambda_1 \langle \mathbf{v}_1, \mathbf{v}_1 \rangle$; $\langle A(\mathbf{v}_1), \mathbf{v}_2 \rangle = B(\mathbf{v}_1, \mathbf{v}_2) = 0 = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle$. Hence

$$\langle A(\mathbf{v}_1) - \lambda_1 \mathbf{v}_1, \mathbf{v}_i \rangle = 0$$

for $i = 1, 2$. Hence $A(\mathbf{v}_1) = \lambda_1 \mathbf{v}_1$.

Let $\mathbf{v} = \sum_{i=1}^n x^i \mathbf{v}_i$, then

$$\begin{aligned} Q(\mathbf{v}) =& B(\mathbf{v}, \mathbf{v}) \\ =& \sum_{i,j=1}^n x^i x^j B(\mathbf{v}_i, \mathbf{v}_j) \\ =& \sum_{i=1}^n \lambda_i (x^i)^2. \end{aligned}$$

□

Let M be a regular orientable surface with unit normal vector field $\mathbf{n}$. Let f be a smooth function on M which is nowhere zero. Let $p \in M$ and let $\mathbf{v}_1$ and $\mathbf{v}_2$ form an orthonormal basis for $T_p(M)$.

(i) Prove that the Gaussian curvature of M at p is given by:

$$K = \frac{\langle d(f\mathbf{n})(\mathbf{v}_1) \times d(f\mathbf{n})(\mathbf{v}_2), \mathbf{n} \rangle}{f^2}.$$

Note $d(f\mathbf{n})(\mathbf{v})$ is defined as follow: let α be the curve on M with $\alpha(0) = p$, $\alpha'(0) = \mathbf{v}$, then

$$d(f\mathbf{n})(\mathbf{v}) = \frac{d}{dt}(f(\alpha(t))\mathbf{n}(\alpha(t)))\Big|_{t=0}.$$

(ii) Let M be the ellipsoid

$$h(x, y, z) := \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Let f be the restriction of the function

$$\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \right)^{\frac{1}{2}}.$$

Apply (i) to show that the Gaussian curvature is given by

$$K = \frac{1}{f^4 a^2 b^2 c^2}.$$

(Hint: We may take $\mathbf{n} = \frac{\nabla h}{|\nabla h|}$. Note that $|\nabla h| = 2f$ and so $d(f\mathbf{n})(\mathbf{v}) = \left(\frac{v_1}{a^2}, \frac{v_2}{v^2}, \frac{v_3}{c^2}\right)$ if $\mathbf{v} = (v_1, v_2, v_3)$.)