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Midterm will be held on October 16, Tuesday 8:30 am–10:15am

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The shape operator and the second fundamental form

1. The shape operator

Let $\mathbf{X} : U \to \mathbb{R}^3$ be a regular surface patch and denote $\mathbf{X}(U)$ by M. Let $\mathbf{N}$ be a *unit normal vector field* on M in the sense that $\mathbf{N} = \mathbf{N}(u, v)$, $(u, v) \in U$ such that

- it is smooth;
- $\mathbf{N}$ has unit length;
- $\mathbf{N}$ is orthogonal to $T_p(M)$ at all point.

Definition 1. The *shape operator* $\mathcal{S}_p$ with respect to $\mathbf{N}$ at p is the operator defined as follows: Let $\mathbf{v} \in T_p(M)$ and let $\alpha(t)$, $-\epsilon < 0 < \epsilon$ be a smooth curve on M with $\alpha(0) = p$. Then $\mathcal{S}_p(\mathbf{v})$ is defined as

$$\mathcal{S}_p(\mathbf{v}) = -\frac{d}{dt}(N(\alpha(t)))\big|_{t=0}.$$

Remark:

- Notice that there is a negative sign on the RHS in the above.
- $\mathcal{S}_p$ is also called the Weingarten map of M at p.
- There are two unit normal vector fields on a regular surface patch, namely $\mathbf{X}_u \times \mathbf{X}_v / |\mathbf{X}_u \times \mathbf{X}_v|$ and its negative.
- If $\mathbf{N}$ is a unit normal vector field, then $\mathbf{N}_1 := -\mathbf{N}$ is also a unit normal vector field. The shape operator with respect to $\mathbf{N}_1$ is the negative of the shape operator with respect to $\mathbf{N}$.

Proposition 1. *With the above notation, the following are true:*

(i) $\mathcal{S}_p$ *is well-defined.*
(ii) $\mathcal{S}_p$ *is a linear map from* $T_p(M)$ *to* $T_p(M)$*.*
(iii) $\mathcal{S}_p$ *is unchanged under reparametrization.*
(iv) $\mathcal{S}_p$ *is self-adjoint with respect to the first fundamental form.*
(v) $\mathcal{S}$ *is smooth.*

Definition 2. Let $\mathcal{S}$ be the shape operator with respect to a unit normal vector field $\mathbf{N}$, the *second fundamental form* $\mathbb{II}_p$ of M at p (with respect to $\mathbf{N}$) is the bilinear form $\mathbb{II}_p(\mathbf{v}, \mathbf{w}) = g(\mathcal{S}_p(\mathbf{v}), \mathbf{w}) = \langle \mathcal{S}_p(\mathbf{v}), \mathbf{w} \rangle$.

Proposition 2. $\mathbb{II}_p$ *is a symmetric bilinear form on* $T_p(M)$*.*

2. Coefficients of the second fundamental form

With the same notation as in the previous section of M. Let $\mathbf{N} = \mathbf{X}_u \times \mathbf{X}_v / |\mathbf{X}_u \times \mathbf{X}_v|$.

Definition 3. The *coefficients of the second fundamental form* e, f, g at p are defined as:

$$\begin{aligned} e =& \mathbb{II}_p(\mathbf{X}_u, \mathbf{X}_u); \\ f =& \mathbb{II}_p(\mathbf{X}_u, \mathbf{X}_v); \\ g =& \mathbb{II}_p(\mathbf{X}_v, \mathbf{X}_v). \end{aligned}$$

Notation: Suppose we use (u^1, u^2) as coordinates, and $\mathbf{N} = \mathbf{X}_1 \times \mathbf{X}_2 / |\mathbf{X}_1 \times \mathbf{X}_2|$, then the coefficients of the second fundamental form are denoted by

$$h_{11} = \mathbb{II}_p(\mathbf{X}_1, \mathbf{X}_1); h_{12} = \mathbb{II}_p(\mathbf{X}_1, \mathbf{X}_2) = h_{21}; h_{22} = \mathbb{II}_p(\mathbf{X}_2, \mathbf{X}_2).$$

Proposition 3.

$$\begin{aligned} e =& \langle \mathbf{N}, \mathbf{X}_{uu} \rangle = \frac{\det(\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu})}{\sqrt{EG - F^2}} \\ f =& \langle \mathbf{N}, \mathbf{X}_{uv} \rangle = \frac{\det(\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{uu})}{\sqrt{EG - F^2}}; \\ g =& \langle \mathbf{N}, \mathbf{X}_{vv} \rangle = \frac{\det(\mathbf{X}_u, \mathbf{X}_v, \mathbf{X}_{vv})}{\sqrt{EG - F^2}}. \end{aligned}$$

3. Gaussian curvature and mean curvature

Suppose $\mathcal{S}_p(\mathbf{X}_u) = a_1^1 \mathbf{X}_u + a_1^2 \mathbf{X}_v, \mathcal{S}_p(\mathbf{X}_v) = a_2^1 \mathbf{X}_u + a_2^2 \mathbf{X}_v$. Then the matrix of S_p with respect to the ordered basis $\beta = \{\mathbf{X}_u, \mathbf{X}_v\}$ is given by

$$[\mathcal{S}_p]_\beta = \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix}$$

Definition 4. The *Gaussian curvature* $K(p)$ of M at p is the determinant of $\mathcal{S}_p$. The *mean curvature* $H(p)$ of M at p is $1/2 \times$the trace of $\mathcal{S}_p$.

Proposition 4. (1) *Let*

$$\begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix}$$

be the matrix of $\mathcal{S}_p$ *with respect to the ordered basis* $\{\mathbf{X}_u, \mathbf{X}_v\}$. *Then*

$$\begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}.$$

(2) *The Gaussian curvature $K(p)$ and the mean curvature $H(p)$ of M at p are given by*

$$K(p) = \frac{eg - f^2}{EG - F^2},$$

and

$$H(p) = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2}.$$

Remark: (i) Gaussian curvature is invariant under reparametrization. (ii) Mean curvature is invariant under *orientation preserving* reparametrization.