More on curvature and torsion

1. Curvature and torsion in general parameter

Proposition 1. *Let $\alpha(t)$ be a regular curve with nonzero curvature. Then the curvature and torsion are given by:*

$$\begin{cases} \kappa = & \frac{|\alpha' \times \alpha''|}{|\alpha'|^3} \\ \tau = & \frac{<\alpha' \times \alpha'', \alpha'''>}{|\alpha' \times \alpha''|^2}. \end{cases}$$

Here $'$ always means differentiation with respect to t.

Proof. Let $\alpha(t)$ be a regular curve with nonzero curvature. Then

$$\alpha' = |\alpha'|T,$$

$$\alpha'' = \kappa|\alpha'|^2 N + |\alpha'|^{-1} < \alpha', \alpha'' > T. \tag{1}$$

Hence

$$< \alpha'', \alpha'' >= \kappa^2 |\alpha'|^4 + |\alpha'|^{-2} < \alpha', \alpha'' >^2,$$

and

$$\begin{aligned} \kappa^2 &= \frac{< \alpha'', \alpha'' >< \alpha', \alpha' > - < \alpha', \alpha'' >^2}{|\alpha'|^6} \\ &= \frac{|\alpha' \times \alpha''|^2}{|\alpha'|^6}. \end{aligned}$$

To compute τ, note that

$$\alpha''' = \kappa(-kT + \tau B)|\alpha'|^3 + f(t)T + g(t)N$$

for some function f and g. (Why?). So

$$\tau = \frac{1}{\kappa} \frac{< \alpha''', B >}{|\alpha'|^3}.$$

Use (1)

$$\begin{aligned} B &= T \times N \\ &= \frac{T \times \alpha''}{k|\alpha'|^2} \\ &= \frac{\alpha' \times \alpha''}{k|\alpha'|^3} \end{aligned}$$

Use the formula for k, we have

$$\tau = \frac{< \alpha' \times \alpha'', \alpha''' >}{|\alpha' \times \alpha''|^2}.$$

□

2. Geometric meaning of curvature

Proposition 2. *Let $\alpha(s)$ be a plane curve parametrized by arc length defined on (a,b). Let $s_0 \in (a,b)$. Suppose $\kappa(s_0) > 0$. Then the following are true:*

(i) *For any $s_1 < s_2 < s_3$ sufficiently close to s_0, $\alpha(s_1), \alpha(s_2), \alpha(s_3)$ are not collinear.*

(ii) *For $s_1 < s_2 < s_3$ sufficiently close to s_0 so that $\alpha(s_1), \alpha(s_2), \alpha(s_3)$ are not collinear, let $c(s_1, s_2, s_3)$ be the center of the unique circle $C(s_1, s_2, s_3)$ passing through $\alpha(s_1), \alpha(s_2), \alpha(s_3)$. As $s_1, s_2, s_3 \to s_0$, $C(s_1, s_2, s_3)$ will converge to a circle passing through $\alpha(s_0)$ tangent to α at $\alpha(s_0)$ with radius $1/\kappa(s_0)$*

Proof. (i) Suppose $\alpha(s_1), \alpha(s_2), \alpha(s_3)$ lie on a straight line. Then

$$\langle \alpha(s_i) - \vec{v}, \vec{n} \rangle = 0$$

for some constant vectors $\vec{v}, \vec{n}$ with $|\vec{n}| = 1$, for $i = 1, 2, 3$. Let $f(s) = \langle \alpha(s) - \vec{v}, \vec{n} \rangle$. Then $f(s_i) = 0$ for $i = 1, 2, 3$. Hence $f'(\xi_1) = f'(\xi_2) = 0$ for some $s_1 < \xi_1 < s_2 < \xi_2 < s_3$ and $f''(\eta) = 0$ for some $\xi_1 < \eta < \xi_2$. That is:

$$\begin{cases} \langle \alpha'(\xi_1), \vec{n} \rangle = & \langle \alpha'(\xi_2), \vec{n} \rangle = 0; \\ \langle \alpha''(\eta), \vec{n} \rangle = & 0. \end{cases}$$

As $s_1, s_2, s_3 \to s_0$, $\vec{n} \to N(s_0)$ and $\alpha''(\eta) = \kappa(s_0) N(s_0)$. This implies $\kappa(s_0) = 0$. Contradiction.

(ii) Let $C(s_1, s_2, s_3)$ be given by

$$||\mathbf{x} - c|| = r.$$

where $c = c(s_1, s_2, s_3)$.

Let $h(s) = ||\alpha(s) - c||^2$. Then $h(s_i) = r^2$ for $i = 1, 2, 3$. Hence $h'(\xi_1) = h'(\xi_2) = 0$ for some $s_1 < \xi_1 < s_2 < \xi_2 < s_3$ and $h''(\eta) = 0$ for some $\xi_1 < \eta < \xi_2$. Hence

$$\begin{cases} \langle \alpha'(\xi_1), \alpha(\xi_1) - c \rangle & = \langle \alpha'(\xi_2), \alpha(\xi_2) - c \rangle = 0; \\ \langle \alpha''(\eta), \alpha(\eta) - c \rangle + 1 & = 0. \end{cases}$$

If $c \to c_\infty$ for some sequence $s_1 < s_2 < s_3 \to s_0$, then

$$\langle \alpha'(s_0), \alpha(s_0) - c_\infty \rangle = 0, \quad \langle \alpha''(s_0), \alpha(s_0) - c_\infty \rangle = -1$$

So $c_\infty - \alpha(s_0) = \frac{1}{\kappa(s_0)} N(s_0)$. From this the result follows. $\square$

The limiting circle is called the *osculating circle*.

Parametrized surface

1. regular parametrized surface patch

Let $U \subset \mathbb{R}^2$ be an open set with coordinates (u^1, u^2).

Definition 1. A *regular parametrized surface patch* or simply a *regular surface* is a map:

$$\mathbf{X} : U \to \mathbb{R}^3$$

such that the following are true:

(rsp1) $\mathbf{X}$ is smooth and injective.
(rsp2) $\mathbf{X}$ is full rank: $\mathbf{X}_1 = \frac{\partial \mathbf{X}}{\partial u^1}$ and $\mathbf{X}_2 = \frac{\partial \mathbf{X}}{\partial u^2}$ are linearly independent, for any $(u^1, u^2) \in U$.

Remarks:

- Let V be an open set in $\mathbb{R}^2$ and let $\phi : V \to U$ be an orientation preserving diffeomorphism. Let $\mathbf{Y} = \mathbf{X} \circ \phi$,

$$\mathbf{Y} : V \to \mathbb{R}^3.$$

  We will not distinguish the two surfaces.
- Condition (rsp2) is equivalent to the following fact that $d\mathbf{X}$ has rank 2 everywhere. This is also equivalent to (i): $\mathbf{X}_1 \times \mathbf{X}_2 \neq \beta 0$ or (ii): one of the following matrices is nonsingular:

$$\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}, \begin{pmatrix} y_1 & y_2 \\ z_1 & z_2 \end{pmatrix}, \begin{pmatrix} z_1 & z_2 \\ x_1 & x_2 \end{pmatrix}.$$

- $\mathbf{X}$ is injective means that the surface has no self intersection.

2. Examples

Proposition 1. *Let $f : U \to \mathbb{R}$ be a smooth function on an open set $U \subset \mathbb{R}^2$. Then the graph of f defined by the following is can be realized as a regular surface:*

$$\mathrm{graph}(f) = \{(x, y, f(x, y)) | \quad (x, y) \in U\}.$$

Example 1: A plane given by the graph of $f(x, y) = ax+by+c$ with constants a, b, c. Or more general: $\mathbf{X}(u^1, u^2) = \mathbf{a}_0 + u^1\mathbf{b}_1 + u^2\mathbf{b}_2$ with $\mathbf{a}_0$, $\mathbf{b}_1, \mathbf{b}_2$ being constant vectors and $\mathbf{b}_1, \mathbf{b}_2$ are linearly independent.

Example 2: (Surface of revolution) Let $x = f(v), z = g(v)$ be a curves in $x{-}z$ plane with $f > 0$, $a < v < b$. $\mathbf{X}(u, v) = (f(v) \cos u, f(v) \sin u, g(v))$.

Example 3 (Ruled surface): Let $\alpha, \mathbf{w} : (a, b) \to \mathbb{R}^3$, be two smooth curves. $\mathbf{X}(t, v) = \alpha(t) + v\mathbf{w}(t)$.

Proposition 2. *Let U be an open set in $\mathbb{R}^3$ and let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Suppose a is a regular value of f. (That is: if $f(\mathbf{x}) = a$, then $\nabla f(\mathbf{x}) \neq \mathbf{0}$.) Then for any $\mathbf{x}_0 = (x_0, y_0, z_0) \in U$ with $f(\mathbf{x}_0) = a$, there is an open set O in $\mathbb{R}^3$ containing $\mathbf{x}_0$ such that $O \cap \{f(\mathbf{x}) = a\}$ can be realized as a regular surface.*

Proof. (Sketch) Let $\mathbf{x}_0 = (x_0, y_0, z_0) \in U$ with $f(\mathbf{x}_0) = a$. Since $\nabla f(\mathbf{x}_0) \neq 0$, we may assume that $f_z(\zeta_0) \neq 0$. Define a map

$$F : U \to \mathbb{R}^3$$

by $F(x, y, z) = (u, v, w)$ with $u = x, v = y, w = f(x, y, z)$. Then at $\mathbf{x}_0$,

$$dF = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x & f_y & f_z \end{pmatrix}$$

which is nonsingular. Now $F(\mathbf{x}_0) = (x_0, y_0, a)$. There are open sets U of $\mathbf{x}_0$ and V of (x_0, y_0, a) such that $F : U \to V$ is a diffeomorphism by the inverse function theorem.

Now

$$F(U \cap \{f = a\}) = \{(u, v, w) \in V |\ w = a\} := O.$$

O can be considered as an open set in $\mathbb{R}^2$. Consider the parametrized surface

$$\mathbf{X} : O \to \mathbb{R}^3$$

with $\mathbf{X}(u, v) = F^{-1}(u, v, a)$. This regular surface and the image is just $U \cap \{f = a\}$.

$\square$

3. Differential and Inverse function theorem

Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map from an open set U to $\mathbb{R}^n$, $F(\mathbf{x}) = \mathbf{y}(\mathbf{x}) =$ where $\mathbf{x} = (x^1, \ldots, x^n)$, $\mathbf{y} = (y^1, \ldots, y^m)$. Let $\mathbf{x_0} = (x_0^1, \ldots, x_0^n) \in U$. The Jacobian matrix of F at $\mathbf{x}_0$ is the $m \times n$ matrix

$$dF_{\mathbf{x}_0} = \left(\frac{\partial y^i}{\partial x^j}(\mathbf{x}_0) \right).$$

Suppose $\mathbf{v}$ is a vector in $\mathbb{R}^n$ represented by a column, then

$$dF_{\mathbf{x}_0}(\mathbf{v}) = \left(\frac{\partial y^i}{\partial x^j}(\mathbf{x}_0) \right) \mathbf{v}.$$

Theorem 1. (Inverse Function Theorem) *Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map. Suppose $F(\mathbf{x}_0) = \mathbf{y}_0$ and $dF_{\mathbf{x}_0}$ is nonsingular. Then there exist open sets $U \supset V \ni \mathbf{x}_0$ and $W \ni \mathbf{y}_0$, such that F is a* *diffeomorphism from V to W. That is to say, $F : V \to W$ is bijective and F^{-1} is also smooth on W.*

Proof. (Sketch) May assume that $\mathbf{x}_0 = \mathbf{0} = \mathbf{y}_0$. Let $A = dF_{\mathbf{x}_0}$. Then

$$F(\mathbf{x}) = A\mathbf{x} + G(\mathbf{x})$$

here $F(\mathbf{x})$ and $\mathbf{x}$ are considered as a column vectors with

$$|G(\mathbf{x}_1) - G(\mathbf{x}_2)| = O(|\mathbf{x}_1 - \mathbf{x}_2|^2)$$

near $\mathbf{0}$. Hence for any $\epsilon > 0$, we can find $\delta > 0$ such that if $\mathbf{x}_1, \mathbf{x}_2 \in B(\mathbf{0}, \delta) = \{|\mathbf{x}| < \delta\}$,

$$|F(\mathbf{x}_1) - F(\mathbf{x}_2)| \ge |A(\mathbf{x}_1 - \mathbf{x}_2)| - \epsilon|\mathbf{x}_1 - \mathbf{x}_2|$$

From this we conclude that F is one-one in $B(\mathbf{0}, \delta)$. (Why?)

Let $\mathbf{b} \in \mathbb{R}^n$. Define

$$\mathbf{a}_0 = A^{-1}\mathbf{b},\ \mathbf{a}_1 = A^{-1}(\mathbf{b} - G(\mathbf{a_0})), \dots,\ \mathbf{a}_{k+1} = A^{-1}(\mathbf{b} - G(\mathbf{a_k})), \dots$$

There is $\rho > 0$, such that if $|\mathbf{b}| < \rho$, the above is well-defined:

- $|\mathbf{a}_0| < \frac{1}{2}\delta$, if $\rho > 0$ is small enough.
- Suppose $|\mathbf{a}_k| < \frac{1}{2}\delta$, then $\mathbf{a}_{k+1}$ can be defined and

  $$|\mathbf{a}_{k+1}| \le C(\rho + |\mathbf{a}_k|^2) \le C(\rho + \frac{1}{4}\delta^2) < \frac{1}{2}\delta$$

  if $\rho > 0, \delta > 0$ are small.

Now

$$|\mathbf{a}_{k+1} - \mathbf{a}_k| \le |A^{-1}G(\mathbf{a}_k - \mathbf{a}_{k-1}) \le C\delta|\mathbf{a}_k - \mathbf{a}_{k-1}|.$$

So $\mathbf{a}_k \to \mathbf{a}$ which is in $B(\mathbf{0}, \delta)$ and $F(\mathbf{a}) = \mathbf{b}$. □

4. Assignment

Assignment 2, Due Friday, 21/9/2018

(1) Show that part of the hyperboloid of one sheet:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

can be realized as a regular surface surface patch with following is a parametrization:

$$\mathbf{X}(u, v) = (a\cosh u \cos v, b\cosh u \sin v, c\sinh u).$$

Find the largest domain on (u, v) plane such that $\mathbf{X}$ is one to one.

(2) Let $\mathbf{S}^1$ be the unit circle $x^2 + y^2 = 1$. Let $\alpha(s), 0 \leq s \leq 2\pi$, be a parametrization of $\mathbf{S}^1$ by arc length. Let $\mathbf{w}(s) = \alpha'(s) + e_3$ where $e_3 = (0, 0, 1)$. Show the ruled surface

$$\mathbf{X}(s, v) = \alpha(s) + v\mathbf{w}(s)$$

with $-\infty < v < \infty$, is part of the hyperboloid $x^2 + y^2 - z^2 = 1$. Is $\mathbf{X}$ a surjective map to the hyperboloid? Is $\mathbf{X}$ injective? Does $\mathbf{X}$ has rank 2 for $0 < s < 2\pi$, $v \in \mathbb{R}$?

(3) Find a parametrization for the catenoid, which is obtained by revolving the catenary $y = \cosh x$ about the x-axis.

(4) The Enneper's surface is defined by

$$\mathbf{X}(u, v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2).$$

Show that this a regular surface patch for $u^2 + v^2 < 3$. Also find two points on the circle $u^2 + v^2 = 3$ such that they have the same image under $\mathbf{X}$.