Theorem 1. *(Revised version) Let $\kappa(s) > 0$ and $\tau(s)$ be smooth function on (a, b). There exists a regular curve $\alpha : (a, b) \to \mathbb{R}^3$ with $|\alpha'| = 1$, such that the curvature and torsion of α are k, τ respectively.*

Moreover, α is unique in the sense that if β is another curve satisfying the above conditions, then $\beta(s) = \alpha(s)P + \vec{c}$ for some constant orthogonal matrix P and some constant vector $\vec{c}$. **Here α, β are considered as row vectors.**

Proof. (**Existence**): Let

$$A(s) = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix}.$$

Let $X(s)$ be the 3×3 matrix and fix s_0 which is the solution of:

$$\begin{cases} X' & = AX \quad \text{in } (a, b); \\ X(s_0) & = I. \end{cases}$$

The solution exists by a theorem in ODE. We claim that X is orthogonal with determinant 1. In fact

$$(X^tX)' = (X^t)'X + X^tX' = (AX)^tX + X^tAX = X^tA^tX + X^tAX = 0$$

because $A^t = -A$. Hence $X^tX = I$ because $X^t(s_0)X(s_0) = I$. (**Using $(XX^t)'$ may be more involved.**) Hence $X(s)$ is orthogonal. Since $\det X(s) = 1$ or -1 and initially, $\det X(s_0) = 1$, we have $\det X(s) = 1$. Write

$$X = \begin{pmatrix} \widetilde{T} \\ \widetilde{N} \\ \widetilde{B} \end{pmatrix}.$$

Define $\alpha(s) = \int_{s_0}^{s} \widetilde{T}(\sigma)d\sigma$. Let T, N, B be the tangent, principal normal and binormal of α, and let $\kappa_\alpha, \tau_\alpha$ be the curvature and torsion of α. Then $\alpha' = \widetilde{T}$ which has length 1. So $T = \widetilde{T}$. Moreover,

$$\kappa_\alpha N = T' = \widetilde{T}' = \kappa \widetilde{N}.$$

we have $\kappa_\alpha = \kappa$ and $N = \widetilde{N}$. Since $\widetilde{T}, \widetilde{N}, \widetilde{B}$ are positively oriented, we conclude that

$$B = T \times N = \widetilde{T} \times \widetilde{N} = \widetilde{B},$$

and

$$-\tau_\alpha N = B' = \widetilde{B}' = -\tau \widetilde{N} = -\tau N.$$

Hence $\tau_\alpha = \tau$.

Uniqness: Let α, β as in the theorem. Let $T_\alpha, N_\alpha, B_\alpha$ be the unit tangent, principal normal, binormal of α; and let $T_\beta, N_\beta, B_\beta$ be the unit tangent, principal normal, binormal of β. Fix $s_0 \in (a, b)$. Let P be an orthogonal matrix with determinant 1 such that

$$\begin{pmatrix} T_\beta(s_0) \\ N_\beta(s_0) \\ B_\beta(s_0) \end{pmatrix} = \begin{pmatrix} T_\alpha(s_0) \\ N_\alpha(s_0) \\ B_\alpha(s_0) \end{pmatrix} P.$$

Here $T_\alpha, \ldots$, etc are considered as row vectors. Let $\gamma(s) = \alpha(s)P$. Let T_γ, N_γ, B_γ be unit tangent, principal normal, binormal of γ. Then

$$T_\gamma = \gamma' = \alpha' P = T_\alpha P,$$

$$\kappa N_\gamma = T'_\gamma = T'_\alpha P = \kappa N P.$$

and so $T_\gamma = T_\alpha P, N_\gamma = N_\alpha P$. Hence $B_\gamma = B_\alpha P$. We have

$$\begin{pmatrix} T_\gamma \\ N_\gamma \\ B_\gamma \end{pmatrix}' = \begin{pmatrix} T_\alpha \\ N_\alpha \\ B_\alpha \end{pmatrix}' P = A \begin{pmatrix} T_\alpha \\ N_\alpha \\ B_\alpha \end{pmatrix} P = A \begin{pmatrix} T_\gamma \\ N_\gamma \\ B_\gamma \end{pmatrix}$$

where A is as above. Since

$$\begin{pmatrix} T_\gamma(s_0) \\ N_\gamma(s_0) \\ B_\gamma(s_0) \end{pmatrix} = \begin{pmatrix} T_\alpha(s_0) \\ N_\alpha(s_0) \\ B_\alpha(s_0) \end{pmatrix} P = \begin{pmatrix} T_\beta(s_0) \\ N_\beta(s_0) \\ B_\beta(s_0) \end{pmatrix}.$$

we have $T_\gamma = T_\beta$, by uniqueness theorem of ODE. So $\gamma(s) + \vec{c} = \beta(s)$ for some constant vector $\vec{c}$. That is: $\beta(s) = \alpha(s)P + \vec{c}$.

□