The Gauss-Bonnet Theorem

1. **Exterior and interior angles of a region**

Definition 1. Let $\alpha : [a, b] \to M$ be a curve. α is said to be *piecewise smooth* if there exist $a = t_0 < t_1 < \cdots < t_k = b$ such that

(i) α is continuous;
(ii) α is regular and is smooth on each $[t_i, t_{i+1}]$.

- α is said to be *simple* if $\alpha(t) \neq \alpha(t')$.
- α is said to be *closed* if $\alpha(a) = \alpha(b)$.
- α is said to be *simple closed* if α is closed and is simple on $(a, b]$.
- α is said to be a *closed geodesic* if α is a smooth geodesic so that $\alpha(a) = \alpha(b)$ and $\alpha'(b) = \alpha'(a)$.

Definition 2. Let M be an oriented regular surface and $\mathcal{R} \subset M$ is a bounded domain in M which is bounded by some piecewise smooth simple closed curve $\alpha_1, \ldots, \alpha_n$. Then α_i is said to be positively oriented if the unit normal $\mathbf{n} \perp \alpha'$ is such that (i) $\alpha', \mathbf{n}$ are positively oriented; and (ii) $\mathbf{n}$ is pointing insider $\mathcal{R}$.

Now let $\mathcal{R} \subset M$ is a bounded domain in M which is bounded by some piecewise smooth positively oriented simple closed curve $\alpha_1, \ldots, \alpha_n$. Denote α be one of the α_k parametrized by arc length with length ℓ. Let $0 = t_0 < t_1 < \cdots < t_{k+1} = \ell$ such that α is smooth on $[t_i, t_{i+1}]$ and α is smooth near $\alpha(0) = \alpha(\ell)$. Each $\alpha(t_i$ is a *vertex*. We want to define *exterior angle* θ_i at $\alpha(t_i)$. First let

$$\alpha'(t_i-) = \lim_{t<t_i, t\to t_i} \alpha'(t); \alpha'(t_i+) = \lim_{t>t_i, t\to t_i} \alpha'(t)$$

- $\alpha(t_i-) = \alpha(t_i+)$, then $\theta_i = 0$.
- $\alpha(t_i-) \neq \pm\alpha(t_i+)$. Then they are linearly independent. Then θ_i is the oriented angle from $\alpha(t_i-)$ to $\alpha(t_i+)$ between $-\pi, \pi$. θ_i is positive (negative), if $\alpha(t_i-), \alpha(t_i+)$ are positively (negatively) oriented.
- $\alpha(t_i-) = -\alpha(t_i+)$, the $\theta_i = \pi$ or $-\pi$. The sign is determined by 'approximation'.

The *interior angle* ι_i at $\alpha(t_i)$ is defined as $\iota_i = \pi - \theta_i$.

2. **Hopf's Umlaufsatz**

Theorem 1 (Hopf's Umlaufsatz, Theorem of Turning Tangents)**.** *Let $\alpha : [0, l] \to \mathbb{R}^2$ be a piecewise regular, simple closed curve. Let $\alpha(t_0)$, $\alpha(t_1), \ldots, \alpha(t_k)$, $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = l$ be the vertices of α with exterior angle θ_i. Let φ_i be smooth choice of angles defined in*

$[t_i, t_{i+1}]$ *such that the oriented angle from the positive axis to* $\alpha'(t)$ *is* $\varphi_i(t)$ *(i.e.* $\alpha' = (\cos\varphi_i(t), \sin\varphi_i(t))$*) for* $t \in [t_i, t_{i+1}]$*. Then*

$$\sum_{i=1}^{k} (\varphi_i(t_{i+1}) - \varphi_i(t_i)) + \sum_{i=0}^{k} \theta_i = \pm 2\pi.$$

Remark 1. Umlauf means "rotation" in German; Umlaufzahl = "rotation number," Satz = "theorem."

3. The Gauss-Bonnet Theorem: local version

Theorem 2. *Let* $\mathbf{X} : U \to M$ *be an isothermal local parametrization of an oriented surface* M *(i.e.* $E = G = e^{2f}$*,* $F = 0$*). Assume that* $\mathbf{X}$ *is orientation preserving. Let* $\alpha = \alpha(s) = \mathbf{X}(u(s), v(s))$*,* $0 \le s \le l$*, be a simple closed curve parametrized by arc length so that* $(u(s), v(s))$ *bounds a region* D *in* U*. Let* $\mathcal{R} = \mathbf{X}(D)$*. Assume* α *is piecewise smooth and positively oriented. Let* $\alpha(s_0), \dots, \alpha(s_k)$ *be the vertices of* α *with exterior angles* $\theta_0, \dots, \theta_k$*, where* $0 = s_0 < s_1 < \cdots < s_k < s_{k+1} = l$*. Then*

$$\int_0^l k_g(s)ds + \iint_{\mathcal{R}} K dA + \sum_{i=0}^{k} \theta_i = 2\pi.$$

We need the following:

Theorem 3. *Let* Ω *be a bounded domain if* $\mathbb{R}^2$ *and let* $\gamma = \gamma(s)$ *parametrized by arc length,* $0 \le s \le l$ *be the boundary curve of* Ω*, positively oriented. Assume that* γ *is piecewise smooth. Let* ν *be the unit outward normal of* Ω*. Suppose* P *and* Q *are two smooth functions defined on* Ω*, and let* $\mathbf{w} = (P, Q)$*. Then*

$$\int_\Omega \left(\frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v}\right) dudv = \int_0^l < \mathbf{w}, \nu > ds.$$

In case $\gamma = \gamma(t)$, $0 \le t \le L$, t may not be arc length. If we write $\gamma(t) = (u(t), v(t))$, $0 \le t \le l$, so that it is positively oriented. Then

$$ds = \sqrt{\left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2} dt.$$

and

$$\nu = \frac{\left(\frac{dv}{dt}, -\frac{du}{dt}\right)}{\sqrt{\left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2}}.$$

Hence

$$\int_0^l < \mathbf{w}, \nu > ds = \int_0^L \left(P\frac{dv}{dt} - Q\frac{du}{dt}\right) dt,$$

and

$$\int_\Omega \left(\frac{\partial P}{\partial u} + \frac{\partial Q}{\partial v} \right) dudv = \int_0^l \left(P\frac{dv}{dt} - Q\frac{du}{dt} \right) dt.$$

Proof of Theorem 2. (Sketch) The geodesic curvature of α is given by

$$k_g = \phi' - u' f_v + v' f_u$$

where $\phi = \phi_i$ is the oriented angle from $\mathbf{X}_u$ to α' on $[s_i, s_{i+1}]$ for each i. The Gaussian curvature is given by

$$K = -e^{2f} \left(f_{uu} + f_{vv} \right).$$

$$\begin{aligned}
\int_0^l k_g(s)ds + \int_{\mathcal{R}} KdA =& \sum_{i=0}^{k} \left(\varphi_i(s_{i+1}) - \varphi_i(s_i) \right) \\
&- \int_0^l (-u' f_v + v' f_u) ds + \int_D \left(f_{uu} + f_{vv} \right) dudv \\
=& \sum_{i=0}^{k} \left(\varphi_i(s_{i+1}) - \varphi_i(s_i) \right) \\
=& 2\pi - \sum_{i=0}^{k} \theta_i
\end{aligned}$$

□

Corollary 1. *Suppose $k = 3$, i.e. we have a triangle then*

$$\int_0^l k_g(s)ds + \iint_{\mathcal{R}} KdA = \sum_{i=1}^{3} \iota_i - \pi,$$

where $\iota_i = \pi - \theta_i$ are the interior angles. Hence if each side is a geodesic, then $K > 0$ implies the sum of the interior angles is larger than π, and $K < 0$, implies the sum of the interior angles is less than π.

4. The Gauss-Bonnet Theorem: global version

Definition 3. A region bounded by a simple closed curved with three vertices (and three edges) is called a triangle. A triangulation of a region $\mathcal{R}$ is a finite family of of triangles $\{T_i\}_{i=1}^n$ such that (i) $\cup_{i=1}^n T_i = \mathcal{R}$; (ii) If $T_i \cap T_j \neq \emptyset$, $i \neq j$, then $T_i \cap T_j$ is a common edge or a common vertex.

Proposition 1. *Every compact region in a regular surface with piecewise smooth boundary has a triangulation so that each triangle is inside an isothermal coordinate neighborhood. If each triangle is positively oriented, then adjacent triangles determine opposite orientation at the common edge.*

Definition 4. Let $\mathcal{R}$ be as in the theorem, and let $\{T_i\}$ be a triangulation. Then $\chi(\mathcal{R}) = V - E + F$, where V = number of vertices; E = number of edges; F = number of faces.

Remark 2. $\chi(\mathcal{R})$ does not depend on the triangulation.

Theorem 4. *Let M be an oriented regular surface and $\mathcal{R}$ is a region in M bounded by piecewise smooth simply closed curve $C_1, \ldots, C_n$ which are positively oriented. Let $\theta_1, \ldots, \theta_l$ be the set of exterior angles of $\mathcal{R}$. Then*

$$\sum_{i=1}^{n} \int_{C_i} k_g ds + \iint_{\mathcal{R}} K dA + \sum_{j=1}^{l} \theta_j = 2\pi\chi(\mathcal{R}),$$

where $\chi(\mathcal{R})$ is the Euler characteristic of $\mathcal{R}$.

Proof of Theorem. (Sketch) Let $\{T_i\}_{i=1}^F$ be a triangulation in the proposition. Let ι_{ik} be the interior angles of T_i. Then

$$\sum_{i=1}^{n} \int_{C_i} k_g ds + \iint_{\mathcal{R}} K dA = \sum_{i=1}^{F} \sum_{k=1}^{3} \iota_{ik} - \pi F.$$

Let E_1=number of external edges, E_2=number internal edges, V_1=number of external vertices, V_2=internal vertices. W_1=number of external vertices which are not end points of and C_i, and W_2=external vertices which are end points of C_i. Note that $W_2 = l$.

$$\sum_{i=1}^{F} \sum_{k=1}^{3} \theta_{ik} = 2\pi V_2 + \pi W_1 + \sum_{j=1}^{l} (\pi - \theta_j).$$

Since $3F = 2E_2 + E_1$, $V_1 = E_1$, we have

$$\begin{aligned}
\sum_{i=1}^{n} \int_{C_i} k_g ds + \iint_{\mathcal{R}} K dA + \sum_{j=1}^{l} \theta_j =& 2\pi V_2 + \pi W_1 + \pi l - \pi F \\
=& 2\pi V_2 + \pi V_1 - \pi F \\
=& 2\pi(F + V) - 3\pi F - \pi V_1 \\
=& 2\pi(F + V) - 2\pi E_2 - \pi E_1 - \pi E_1 \\
=& 2\pi(V - E + F).
\end{aligned}$$

□

Let M be a compact oriented surface. Then M is diffeomorphic to the unit sphere $\mathbb{S}^2$ with g handles attached, (g is called the genus of M). Moreover, $\chi(M) = 2 - 2g$.

Corollary 2. *Let M be a regular orientable compact surface. Then*

$$\iint_M K dA = 2\pi\chi(M) = 4\pi(1-g),$$

where g is the genus of M. Hence: (i) $\int_M K dA > 0$ if and only if M is diffeomorphic to $\mathbb{S}^2$; (ii) $\int_M K dA = 0$ if and only if M is diffeomorphic to the torus; and (iii) $\int_M K dA < 0$ if and only if M is diffeomorphic to $\mathbb{S}^2$ with g handles attached for some $g \geq 2$.

Theorem 5 (The Jordan curve theorem)**.** *Every simple closed curve in $\mathbb{R}^2$ will divide $\mathbb{R}^2$ into exactly two components.*

Assignment 10, Due Friday Nov 30

(1) Let M be a compact oriented regular surface. Find the $\chi(M)$ if (i) M is homeomorphic to a sphere; (ii) M is homoeomorphic to a torus. Also find $\int_M K dA$ in each case.

(2) Prove that if M is a regular surface with nonpositive Gaussian curvature. Prove that two geodesics from a point p cannot meet again so that they form the boundary of a simple region, i.e. an open set which is homeomorphic to a disk.

(3) Let M be compact surface with positive Gaussian curvature. Prove that M is homeomorphic to a sphere. Prove also that if there exist two distinct simple closed geodesics on M, then they must intersect.