1. **Geodesic curvature on an orientable surface**

Definition 1. Let M be a regular surface. M is said to be *orientable* if there exist coordinate charts $\mathcal{A} = \{(\mathbf{X}^{(k)}, U_k)\}$ covering M such that $\mathbf{X}^{(k)} \circ (\mathbf{X}^{(j)})^{-1}$ (if defined) is orientation preserving: the Jacobian matrix has positive determinant.

- Suppose M is orientable with coordinate charts $\mathcal{A}$ as in the definition. For any $p \in M$ and let $\mathbf{v}_1, \mathbf{v}_2$ be linearly independent vectors in $T_p(M)$. $\mathbf{v}_1, \mathbf{v}_2$ are said to be *positively oriented* if and only if for any k with $p \in \mathbf{X}^{(k)}(U_k)$ the transformation matrix from $\mathbf{v}_1, \mathbf{v}_2$ to $\mathbf{X}_1^{(k)}, \mathbf{X}_2^{(k)}$ has positive determinant.
- Suppose M is orientable. Then a choice of coordinate charts $\mathcal{A}$ as in the definition is called an *orientation* of M. There are 'basically' two orientations of an orientable surface.
- Given an orientation $\mathcal{A}$, let us denote $-\mathcal{A}$ to be the opposite orientation.
- Suppose M is orientable with orientation $\mathcal{A}$. Let α be a regular curve on M parametrized by arclength. Let $\mathbf{n} \in T_p(M)$ be such that $\alpha', \mathbf{n}$ are positively oriented. Then the geodesic curvature is given by $k_g = \langle \alpha'', \mathbf{n} \rangle$.
- The geodesic curvature will change sign if the orientation of M is changed to the opposite orientation.
- Fix an orientation in M. Suppose the orientation of α is changed to its 'negative' then the geodesic curvature will change sign.

2. **Geodesics of surfaces of revolution**

Let $(\phi(v), 0, \psi(v))$ be a regular curve on the xz-plane such that:

(i) $\phi(v) > 0$, i.e. the curve does not intersect the z-axis.
(ii) $(\phi_v)^2 + (\psi_v)^2 = 1$, i.e. the curve is parametrized by arc length.

Consider the surface of revolution M given by

$$\mathbf{X}(u, v) = (\phi(v) \cos u, \phi(v) \sin u, \psi(v)).$$

- The curve $\mathbf{X}(u_0, v)$ where u_0 is a constant is called a *meridian*; and
- the curve $\mathbf{X}(u, v_0)$ where v_0 is a constant is called a *parallel.*
- The first fundamental form is given by:

$$\begin{cases} g_{11} = & E = \langle \mathbf{X}_u, \mathbf{X}_u \rangle = \phi^2, ; \\ g_{12} = & g_{21} = F = \langle \mathbf{X}_u, \mathbf{X}_v \rangle = 0 \\ g_{22} = & G = \langle \mathbf{X}_v, \mathbf{X}_v \rangle = (\phi_v)^2 + (\psi_v)^2 = 1. \end{cases}$$

- The Christoffel symbols are: $\Gamma^1_{12} = \Gamma^1_{21} = \phi_v/\phi$, $\Gamma^2_{11} = -\phi\phi_v$ and all other Γ^k_{ij} are zeros.
- $\alpha(t) = \mathbf{X}(u(t), v(t))$ is a geodesic if and only if

$$\begin{cases} u'' + \dfrac{2\phi_v}{\phi} u'v' = & 0, \\ v'' - \phi\phi_v (u')^2 = & 0. \end{cases}$$

Corollary 1. *Any meridian is a geodesic. A parallel $\mathbf{X}(u, v_0)$ is a geodesic if and only if $\phi_v(v_0) = 0$.*

To study the behavior of general geodesics, we begin with the following lemma:

Lemma 1. *Let $a_1(t), a_2(t)$ be smooth functions on $(T_1, T_2) \subset \mathbb{R}$ such that $a_1^2 + a_2^2 = 1$. For any $t_0 \in (T_1, T_2)$ and θ_0 such that $a_1(t_0) = \cos\theta_0$, $a_2(t_0) = \sin\theta_0$, there exists unique a smooth function $\theta(t)$ with $\theta(t_0) = \theta_0$ such that $a_1(t) = \cos\theta(t)$ and $a_2(t) = \sin\theta(t)$.*

Proof. (Sketch) Suppose θ satisfies the condition. Then $a_1' = -\theta' \sin\theta$, $a_2' = \theta' \cos\theta$. Hence $\theta' = a_1 a_2' - a_2 a_1'$. From this we have uniqueness. To prove existnce, fix $t_0 \in (T_1, T_2)$ and let θ_0 be such that $\cos\theta_0 = a_1(0)$, $\sin\theta_0 = a_2(0)$. Let

$$\theta(t) = \theta_0 + \int_{t_0}^{t} (a_1 a_2' - a_2 a_1') d\tau.$$

Let $f = (a_1 - \cos\theta)^2 + (a_2 - \sin\theta)^2$, with then

$$\begin{aligned} f' =& 2(a_1 - \cos\theta)(a_1' + \sin\theta\theta') + 2(a_2 - \sin\theta)(a_2' - \cos\theta\theta') \\ =& 2\left(a_1 a_1' + a_2 a_2' - a_1' \cos\theta - a_2' \sin\theta + \theta'(a_1 \sin\theta - a_2 \cos\theta)\right) \\ =& 2\left(-a_1' \cos\theta - a_2' \sin\theta + (a_1 a_2' - a_2 a_1')(a_1 \sin\theta - a_2\cos\theta)\right) \\ =& 0 \end{aligned}$$

because $a_1^2 + a_2^2 = 1$. So θ is a smooth function and is a required function. $\square$

Now let $\alpha(s) = \mathbf{X}(u(s), v(s))$ be a geodesic on M parametrized by arc length. Let $\mathbf{e_1} = \mathbf{X}_u/|\mathbf{X}_u|$ and $\mathbf{e_2} = \mathbf{X}_v/|\mathbf{X}_v|$. Then $\mathbf{e}_1, \mathbf{e}_2$ are orthonormal. Let

$$\alpha' = a_1 \mathbf{e_1} + a_2 \mathbf{e_2}.$$

By the lemma there exists smooth function $\theta(s)$ such that $a_1 = \sin\theta$, $a_2 = \theta$. Note that θ *is the angle between α' and the meridian.*

Proposition 1 (Clairaut's Theorem)**.** *$r(s)\sin\theta(s)$ is constant along α, where $r(s)$ is the distance of $\alpha(s)$ from the z-axis.*

Proof. (Sketch) Denote the $\frac{d\alpha}{ds}$ by α' etc. Since $r(s) = \phi(v(s))$,

$$r' = \phi_v v'.$$

Also $\sin\theta = \langle \alpha', \mathbf{e_1} \rangle = u'\phi$, so $(\sin\theta)' = u''\phi + u'v'\phi_v$.

$$\begin{aligned}(r\sin\theta)' =& \phi_v v' u'\phi + u''\phi + \phi_v u' v' \\ =& \phi\left(u'' + \frac{2\phi_v}{\phi}u'v'\right) \\ =& 0.\end{aligned}$$

□

Let us analyse a geodesic $\alpha(s)$, $0 \le s < L \le \infty$, on the surface of revolution parametrized by arc length. Let us assume that $\psi(v)$ is increasing. Let $r(s)$ and $\theta(s)$ be as in Clairaut's Theorem. Let $\theta_0 = \theta(0)$. We may assume that $0 \le \theta_0 \le \frac{\pi}{2}$. By the theorem, $r(s)\sin\theta(s) = R$ for some constant $R \ge 0$. Note that $r(s) \ge R$.

Proposition 2. (i) *If $R = 0$, then α is a meridian.*

(ii) *$R > 0$. Then geodesic will go up for all s, as long as $r > R$, i.e. the z coordinate of α is increasing in s. Either α does not come close to any parallel of radius R, and α will go up for all s, or α will be close to a parallel C of radius R. Let C be the first such parallel above α. Then we have the following cases:*
   - (a) *C is a geodesic. Then α will not meet C and α will come arbitrarily close to C without intersecting C.*
   - (b) *C is not a geodesic. Then there is $\alpha(s_0) \in C$ for some s_0 and α will bounce off from C and will turn downward.*

Assignment 9, Due Friday Nov 23

(1) Let M be a regular surface in $\mathbb{R}^3$ with a continuous normal vector field $\mathbf{N}$. Show that M is orientable as in the Definition 1.

(2) Write down the differential equations for the geodesics on the torus:

$$\mathbf{X}(u,v) = ((a + r\cos v)\cos u, (a + r\cos v)\sin u, r\sin v)$$

with $a > r > 0$.

Also, show that if α is a geodesic start at a point on the top-most parallel $(a\cos u, a\sin u, r)$ and is tangent to this parallel, then α will stay in the region with $-\pi/2 \le v \le \pi/2$.

(3) Let M be an orientable regular surface. Let $\mathbf{X} : U \to M$, $(u_1, u_2) \to \mathbf{X}(u_1, u_2)$, be a coordinate parametization, with U being an open set in $\mathbb{R}^2$ so that $\mathbf{X}_1, \mathbf{X}_2$ are positively oriented.

Suppose the first fundamental form in this coordinate is such that $g_{ij} = \exp(2f)\delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and is zero if $i \neq j$. $\mathbf{e_1} = \mathbf{X}_1/|\mathbf{X}_1|$, $\mathbf{e_2} = \mathbf{X}_2/|\mathbf{X}_2|$, and $\mathbf{n}$ be such that $\alpha', \mathbf{n}$ are positively oriented. Let $\alpha(s)$ be a geodesic on M such that $\alpha(s) = \mathbf{X}(u_1(s), u_2(s))$. Let $\theta(s)$ be a smooth function on s such that $\alpha'(s) = \mathbf{e_1}(s)\cos\theta(s) + \mathbf{e_2}(s)\sin\theta(s)$, where $\mathbf{e_i}(s) = \mathbf{e_i}(\alpha(s))$. Show that

$$k_g == \left(-u'\frac{\partial f}{\partial v} + v'\frac{\partial f}{\partial u}\right) + \theta'.$$

(Note that if $f = 1$, i.e. M is a plane, then $k_g = \theta'$.)