1. **Further properties of Geodesics**

Proposition 1. *Isometry will carry geodesics to geodesics.*

Theorem 1. *At any point $p \in M$, and any vector $\mathbf{v} \in T_p(M)$, there is a geodesic $\alpha(t)$ defined on $(-\epsilon, \epsilon)$ for some $\epsilon > 0$ such that $\alpha(0) = p$ and $\alpha'(0) = \mathbf{v}$.*

The theorem follows from the following theorem in ODE.

Theorem 2. *Let U be an open set in $\mathbb{R}^n$ and let $I_a = (-a, a) \subset \mathbb{R}$, with $a > 0$. Suppose $\mathbf{F} : U \times I_a \to \mathbb{R}^n$ is a smooth map. Then for any $\mathbf{x}_0 \in U$, there is $0 < \delta < a$, such that the following* **IVP** *has a solution:*

$$\begin{cases} \mathbf{x}'(t) = & \mathbf{F}(x(t), t), \ -\delta < t < \delta; \\ \mathbf{x}(0) = & \mathbf{x}_0. \end{cases}$$

Moreover, the solutions of the **IVP** *is unique. Namely, if $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions of the above* **IVP** *on $(-b, b)$ for some $0 < b < a$, then $\mathbf{x}_1 = \mathbf{x}_2$.*

2. **The energy of a curve**

Let M be a regular surface and α be a smooth curve defined on $[a, b]$. Then *energy* of α is defined to by

$$E(\alpha) = \frac{1}{2} \int_a^b \langle \alpha', \alpha' \rangle dt. \tag{1}$$

$\langle \alpha', \alpha' \rangle$ is called the energy density.

Remark: With the above notation, $(\ell(\alpha))^2 \leq (b - a)E(\alpha)$, and equality holds if and only if α is parametrized proportional to arc length.

Theorem 3. *Suppose α is a smooth curve defined on $[a, b]$. α is an extremal of E if and only if α is a geodesic.*

Example: Consider the surface of revolution $u^1 \leftrightarrow u$, $u^2 \leftrightarrow v$:

$$\mathbf{X}(u, v) = (f(v) \cos u, f(v) \sin u, g(v))$$

$f > 0$. We want to find the equations of geodesics.

Method 1: $g_{11} = f^2$, $g_{12} = 0$, $g_{22} = (f')^2 + (g')^2$. The Christoffel symbols are given by

$$\Gamma_{11}^1 = 0, \Gamma_{11}^2 = -\frac{ff'}{(f')^2 + (g')^2}, \Gamma_{12}^1 = \frac{ff'}{f^2};$$

$$\Gamma_{12}^2 = 0, \Gamma_{22}^1 = 0, \Gamma_{22}^2 = \frac{f'f'' + g'g''}{(f')^2 + (g')^2}.$$

Hence geodesic equations are

$$\ddot{u} + \frac{2ff'}{f^2}\dot{u}\dot{v} = 0$$

and

$$\ddot{v} - \frac{ff'}{(f')^2 + (g')^2}\dot{u}^2 + \frac{f'f'' + g'g''}{(f')^2 + (g')^2}\dot{v}^2 = 0.$$

Method 2: On the other hand, the energy density of a curve is given by

$$\mathcal{L} = \frac{1}{2}(f^2(\dot{u})^2 + ((f')^2 + ((g')^2)(\dot{v})^2).$$

Then

$$\frac{\partial}{\partial u}\mathcal{L} = 0, \frac{\partial}{\partial v}\mathcal{L} = ff'\dot{u}^2 + (f'f'' + g'g'')\dot{v}^2;$$
$$\frac{\partial}{\partial \dot{u}}\mathcal{L} = f^2\dot{u}, \frac{\partial}{\partial \dot{v}}\mathcal{L} = ((f')^2 + ((g')^2)\dot{v}.$$

The E-L equations are the geodesic equations.

Assignment 8, Due Friday, November 16

(1) (a) Find the absolute value of the curvature of the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the points $(a, 0)$ and $(0, b)$. Assuming $a, b > 0$.

(b) Intersect the cylinder $C = \{(x, y, z)|x^2 + y^2 = 1\}$ with a plane passing through the x-axis and making an angle θ with the xy-plane. Show that the curve α is an ellipse. Also find the absolute value of the geodesic curvature of α at the points where α meets their axes (i.e. major and minor axes of the ellipse).

(2) Prove Theorem 3: A regular curve $\alpha(t) = \mathbf{X}(u^1(t), u^2(t))$ in a regular surface is a geodesic if and only if $u^1(t), u^2(t)$ satisfy the system of geodesic equations.

(3) Let α be a regular curve in a regular surface such that $\alpha'' \neq 0$. Suppose α is a geodesic and α is contained in a plane. Prove that α' is a principal direction.

(4) Let $p \in M$ be a point in a regular surface. Suppose any geodesic passing through p is a plane curve. (Different geodesics may be contained in different planes). Prove that p is an umbilical point. What can you say if every point of M satisfies this property?

$$
\begin{aligned}
\alpha'' =& u_1''\mathbf{X}_1 + u_2''\mathbf{X}_2 + (u_1')^2\mathbf{X}_{11} + 2u_1'u_2'\mathbf{X}_{12} + (u_2')^2\mathbf{X}_{22}) \\
=& \mathbf{X}_1\left(u_1'' + \Gamma_{11}^1(u_1')^2 + 2\Gamma_{12}^1 u_1'u_2' + \Gamma_{22}^1(u_2')^2\right) \\
& + \mathbf{X}_2\left(u_2'' + \Gamma_{11}^2(u_1')^2 + 2\Gamma_{12}^2 u_1'u_2' + \Gamma_{22}^2(u_2')^2\right) \\
& + c\mathbf{n} \\
=& \sum_{k=1}^{2}\mathbf{X}_k\left(u_k'' + \sum_{i,j=1}^{2}\Gamma_{ij}^k u_i'u_j'\right) + c\mathbf{n}
\end{aligned}
$$

$$
\begin{aligned}
k_g =& \frac{\langle \alpha' \times \alpha'', \mathbf{X}_1 \times \mathbf{X}_2\rangle}{\sqrt{\det(g_{ij})}} \\
=& \frac{1}{\sqrt{\det(g_{ij})}}\left(\langle \alpha', \mathbf{X}_1\rangle\langle \alpha'', \mathbf{X}_2\rangle - \langle \alpha', \mathbf{X}_2\rangle\langle \alpha'', \mathbf{X}_1\rangle\right) \\
=& \frac{1}{\sqrt{\det(g_{ij})}}\Bigg[(g_{11}u_1' + g_{12}u_2')\left(g_{12}\left(u_1'' + \sum_{i,j=1}^{2}\Gamma_{ij}^1 u_i'u_j'\right) + g_{22}\left(u_2'' + \sum_{i,j=1}^{2}\Gamma_{ij}^2 u_i'u_j'\right)\right) \\
& - (g_{12}u_1' + g_{22}u_2')\left(g_{11}\left(u_1'' + \sum_{i,j=1}^{2}\Gamma_{ij}^1 u_i'u_j'\right) + g_{12}\left(u_2'' + \sum_{i,j=1}^{2}\Gamma_{ij}^2 u_i'u_j'\right)\right)\Bigg] \\
=& \sqrt{\det(g_{ij})}\left[u_1'\left(u_2'' + \sum_{i,j=1}^{2}\Gamma_{ij}^2 u_i'u_j'\right) - u_2'\left(u_1'' + \sum_{i,j=1}^{2}\Gamma_{ij}^1 u_i'u_j'\right)\right] \\
=& \sqrt{\det(g_{ij})}\Big[u_1'u_2'' - u_2'u_1'' + \Gamma_{11}^2(u_1')^3 - \Gamma_{22}^1(u_2')^3 + \left(2\Gamma_{12}^2 - \Gamma_{11}^1\right)(u_1')^2u_2' \\
& - \left(2\Gamma_{12}^1 - \Gamma_{22}^2\right)(u_2')^2u_1'\Big]
\end{aligned}
$$

More intrinsic: Let $e_1 = \alpha'$ and $e_2 = \mathbf{n}$. Then

$$e_2 = a\mathbf{X}_1 + b\mathbf{X}_2$$

$|e_2| = 1, \langle e_1, e_2\rangle = 0$ implies that

$$a^2E + 2abF + b^2G = 1; \quad a\langle\alpha', \mathbf{X}_1\rangle + b\langle\alpha', \mathbf{X}_2\rangle = 0.$$

Let $\lambda = \langle\alpha', \mathbf{X}_1\rangle, \mu = \langle\alpha', \mathbf{X}_2\rangle$. Assume $\mu \neq 0$, then

$$b = -\frac{\lambda}{\mu}a.$$

Hence

$$a^2(E - 2\frac{\lambda}{\mu}F + \frac{\lambda^2}{\mu^2}G) = 1$$

So

$$a^2 = \frac{\mu^2}{E\mu^2 - 2\lambda\mu F + \lambda^2 G}$$

Let v be any vector, then

$$\begin{aligned}
\langle v, e_2\rangle =& a\langle v, \mathbf{X}_1\rangle + b\langle v, \mathbf{X}_2\rangle \\
=& a\langle v, \mathbf{X}_1\rangle - \frac{\lambda}{\mu}a\langle v, \mathbf{X}_2\rangle \\
=& \frac{a}{\mu}\left(\mu\langle v, \mathbf{X}_1\rangle - \lambda\langle v, \mathbf{X}_2\rangle\right) \\
=& \frac{a}{\mu}\left(\langle\alpha', \mathbf{X}_2\rangle\langle v, \mathbf{X}_1\rangle - \langle\alpha', \mathbf{X}_1\rangle\langle v, \mathbf{X}_2\rangle\right) \\
=& \pm\frac{1}{(E\mu^2 - 2\lambda\mu F + \lambda^2 G)^{\frac{1}{2}}}\left(\langle\alpha', \mathbf{X}_2\rangle\langle v, \mathbf{X}_1\rangle - \langle\alpha', \mathbf{X}_1\rangle\langle v, \mathbf{X}_2\rangle\right)
\end{aligned}$$

If $\mathbf{X}_1 = \alpha e_1 + \beta e_2$, $\mathbf{X}_2 = \gamma e_1 + \delta e_2$. Then

$$\begin{aligned}
E\mu^2 - 2\lambda\mu F + \lambda^2 G =& |\mathbf{X}_1|^2\langle e_1, \mathbf{X}_2\rangle^2 - 2\langle\mathbf{X}_1, \mathbf{X}_2\rangle\langle e_1, \mathbf{X}_2\rangle\langle e_1, \mathbf{X}_1\rangle + |\mathbf{X}_2|^2\langle e_1, \mathbf{X}_1\rangle^2 \\
=& (\alpha^2 + \beta^2)\gamma^2 + (\gamma^2 + \delta^2)\alpha^2 - 2\alpha\gamma(\alpha\gamma + \beta\delta) \\
=& \beta^2\gamma^2 + \alpha^2\delta^2 - 2\alpha\beta\gamma\delta \\
=& (\alpha\delta - \beta\gamma)^2 \\
=& EG - F^2.
\end{aligned}$$