MAT 4030: DIFFERENTIAL GEOMETRY I
2018-19, 1st term

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References:

Oprea: *Differential geometry and its applications* ;

Do Carmo: *Differential geometry of curves and surfaces*;

Klingenberg: *A course in Differential Geometry*;

Spivak: *A comprehensive introduction to Differential Geometry, Vol. 2*

Hilbert and Cohbn-Vossen: *Geometry and the imagination.*

(We will basically follow the book by Oprea.)

Assessment Scheme: Homework 10%; Midterm 30%, Final Exam 60%.

Assignment 1, Due 13/9/2018

(1) (The tractrix) Let $\alpha : (0, \pi) \to \mathbb{R}^2$ be given by

$$\alpha(t) = \left(\sin t, \cos t + \log \tan \frac{t}{2} \right).$$

(a) Prove that α is regular except at $t = \frac{\pi}{2}$.

(b) Prove that the length of the segment of the tangent of α between the point of tangency and the y-axis is constantly 1.

(2) A regular curve $\alpha(s)$ parametrized by arc length is called a *cylindrical helix* if the is some constant vector $\mathbf{u}$ such that $\langle T, \mathbf{u} \rangle = \cos \theta_0$ is a constant. Prove that a regular curve α parametrized by arc length with $\kappa > 0$ is a cylindrical helix if and only if κ/τ is constant.

(3) Assume that $k(s) > 0$, $\tau(s) \neq 0$ and $k'(s) \neq 0$ for all s for a regular curve $\alpha(s)$ parametrized by arc length. Show that α lies on a sphere if and only if

$$\rho^2 + (\rho')^2 \lambda^2 = \text{constant}.$$

where $\rho = 1/k(s)$, $\lambda = 1/\tau$.

(*Hint*: Necessity: Differentiate $|\alpha|^2$ three times to obtain $\alpha = -\rho N - \rho' \lambda B$. Sufficiency: Show that $\beta = \alpha + \rho N - \rho' \Gamma B$ is constant.)

1. **The Frenet formula**

Let $\alpha(s)$ be the regular curve parametrized by arc length.
Let $\vec{T} = \alpha'$. Then

$$\begin{aligned} k(s) :=& |T'|(s) \quad \text{(curvature)};\\ N(s) :=& \frac{1}{k(s)} T'(s) \quad \text{(normal, if } k > 0);\\ B(s) :=& T(s) \times N(s) \quad \text{(binormal, if } k > 0). \end{aligned}$$

Fact: $B' = -\tau N$, τ is called the torsion of α.

Theorem 1. *(Frenet formula) Let α be a regular curve with curvature $k > 0$. Then*

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

We summarize some properties on curves:

Theorem 2. *Let α be a regular curves in $\mathbb{R}^3$ parametrized by arc length.*

- (i) *Suppose the curvature $k \equiv 0$ if and only if α is a straight line.*
- (ii) *Suppose the curvature $k > 0$ and the torsion $\tau \equiv 0$ if and only if α is a plane curve.*
- (iii) *Suppose the curvature $k = k_0 > 0$ is a constant and $\tau \equiv 0$, then α is a circular arc with radius $1/k_0$.*
- (iv) *Suppose the curvature $k > 0$ and the torsion $\tau \neq 0$ everywhere. α lies on s sphere if and only if $\rho^2 + (\rho')^2\lambda^2 =$constant, where $\rho = 1/k$ and $\lambda = 1/\tau$.*
- (v) *Suppose the curvature $k = k_0 > 0$ is a constant and $\tau = \tau_0$ is a constant. Then α is a circular helix.*
- (vi) *Suppose α is defined on $[a, b]$. Let $\mathbf{p} = \alpha(a)$ and $\mathbf{q} = \alpha(b)$. Then the length l of α satisfies $l \leq |\mathbf{p} - \mathbf{q}|$. Moreover, equality holds if and only if α is the straight line from $\mathbf{p}$ to $\mathbf{q}$.*

2. **Some results of the local theory of curves**

Theorem 3. *Let $k(s) > 0$ and $\tau(s)$ be smooth function on (a, b). There exists a regular curve $\alpha : (a, b) \to \mathbb{R}^3$ with $|\alpha'| = 1$, such that the curvature and torsion of α are k, τ respectively.*

Moreover, α is unique in the sense that if β is another curve satisfying the above conditions, then $\beta(s) = A\alpha(s) + \vec{c}$ for some constant orthogonal matrix A and some constant vector $\vec{c}$.

Theorem 4. *Let $\alpha : I \to \mathbb{R}^3$ be a regular curve parametrized by arc-length so that $k > 0$. Assume $0 \in I$ and $T(0) = e_1, N = e_2, B = e_3$, where $\{e_i\}$ is the standard ordered basis for $\mathbb{R}^3$. Then for $|s|$ small, $\alpha(s) = (x(s), y(s), z(s))$ is given by*

$$\begin{cases} x(s) = s - \frac{1}{6}k^2 s^3 + O(|s|^4) \\ y(s) = \frac{1}{2}ks^2 + \frac{1}{6}k' s^3 + O(|s|^4) \\ z(s) = \frac{1}{6}k\tau s^3 + O(|s|^4). \end{cases}$$

3. Existence and uniqueness thoerems in ODE

Ref: *Ordinary differential equations, Birkoff and Rota*

We only consider the special case of linear ODE. Let $A(t) = (a_{ij}(t))_{n\times n}$ be a smooth family $n \times n$ matrix, $t \in [a, b]$. Consider the following initial valued problem (IVP): Given A and a constant $\mathbf{x}_0 \in \mathbb{R}^n$, to find $\mathbf{x} : [a, b] \to \mathbb{R}^n$ satisfying:

$$\begin{cases} \mathbf{x}'(t) & = A(t)\mathbf{x}(t), \quad t \in [a, b]; \\ \mathbf{x}(a) & = \mathbf{x}_0. \end{cases}$$

Theorem 5. *Given any $\mathbf{x}_0 \in \mathbb{R}^n$, the exists a unique solution of the above IVP.*

Proof. (Sketch) For simplicity let us assume $a = 0$.

Existence: Define inductively, with $\mathbf{x}_0(t) = \mathbf{x}_0$ for all t, and

$$\mathbf{x}_{k+1}(t) = \mathbf{x}_0 + \int_0^t A(\tau)\mathbf{x}_k(\tau)d\tau.$$

for $k \geq 0$. Let $M = \sup_{t\in[a,b]} ||A||(t)$ and $||A(t)||^2 = \mathrm{tr}(\mathrm{AA^T(t)})$. For $k \geq 1$, we have

$$|\mathbf{x}_{k+1}(t) - \mathbf{x}_k(t)| \leq M \int_0^t |\mathbf{x}_k(\tau) - \mathbf{x}_{k-1}(\tau)|d\tau.$$

Inductively, we have (why?)

$$\begin{aligned} |\mathbf{x}_{k+1}(t) - \mathbf{x}_k(t)| \leq & M^k \int_0^t \int_0^{\tau_{k-1}} \cdots \int_0^{\tau_2} \int_0^{\tau_1} |\mathbf{x}_1(\tau_1) - \mathbf{x}_0(\tau_1)| d\tau_1 d\tau_2 \dots d\tau_{k-1} d\tau_k \\ \leq & \frac{M^k b^k S}{k!} \end{aligned}$$

where integration is over the domain $t \geq \tau_k \geq \cdots \geq \tau_1$ and $S = \sup_{t\in[0,b]} |\mathbf{x}_1(t) - \mathbf{x}_0(t)|$.

Hence $\sum_{k=1}^{\infty} |\mathbf{x}_{k+1}(t) - \mathbf{x}_k(t)| \leq C$ for some constant C for all $t \in [0, b]$. This implies that $\mathbf{x}_k \to \mathbf{x}_\infty$ uniformly on $[0, b]$ which satisfies:

$$\mathbf{x}_\infty(t) = \mathbf{x}_0 + \int_0^t A(\tau)\mathbf{x}_\infty(\tau) d\tau,$$

(why?) Now $\mathbf{x}_\infty$ is the solution of the above IVP.

Uniquess: Sufficient to prove that if $\mathbf{x}_0 = \mathbf{0}$, then any solution must be trivial. So let $\mathbf{x}$ be such a solution, then

$$\frac{d}{dt}||\mathbf{x}||^2 = 2\langle A\mathbf{x}, \mathbf{x}\rangle \leq 2M||\mathbf{x}||^2.$$

Hence

$$\frac{d}{dt}\left(\exp(-2Mt)||\mathbf{x}||^2\right) \leq 0.$$

This will imply that $||\mathbf{x}||^2 \equiv 0$. (Why?) $\square$