Covariant derivatives

1. Vector fields

In the following we will use Einstein summation convention.

In a coordinate chart with coordinates $x^1, \dots, x^n$, let $\frac{\partial}{\partial x^i}$ be the vector field generated by the curves $\{x^j = \text{constant}, \forall j \neq i\}$. Then any vector field V can be expressed as

$$V = a^i \frac{\partial}{\partial x^i}.$$

If $y^1, \dots, y^n$ are another coordinates, then

$$\frac{\partial}{\partial y^i} = \frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^k}; \frac{\partial}{\partial x^i} = \frac{\partial x y^k}{\partial x^i}\frac{\partial}{\partial y^k}.$$

Note that

$$\frac{\partial x^k}{\partial y^i}\frac{\partial y^i}{\partial x^l} = \delta^k_l.$$

In y coordinates, V can be expressed as

$$V = b^i \frac{\partial}{\partial y^i}.$$

Then

$$V = b^i \frac{\partial}{\partial y^i} = b^i \frac{\partial x^k}{\partial y^i}\frac{\partial}{\partial x^k} = a^k \frac{\partial}{\partial x^k}.$$

Hence a^i, b^j are related by

$$b^i \frac{\partial x^k}{\partial y^i} = a^k;\ b^i = \frac{\partial y^i}{\partial x^k} a^k.$$

In particular,

$$\frac{\partial}{\partial x^i} = \frac{\partial y^k}{\partial x^i}\frac{\partial}{\partial y^k}$$

etc.

We want to differentiate V along $\frac{\partial}{\partial x^i}$. Let us denote this by $D_{\frac{\partial}{\partial x^i}} V$. The most obvious way is to let

$$D_{\frac{\partial}{\partial x^i}} V = \frac{\partial a^j}{\partial x^i}\frac{\partial}{\partial x^j}$$

But if we use the y coordinates, we might want

$$D_{\frac{\partial}{\partial y^i}} V = \frac{\partial b^j}{\partial y^i}\frac{\partial}{\partial y^j}.$$

But

$$\begin{aligned}
D_{\frac{\partial}{\partial x^i}} V =& D_{\frac{\partial y^k}{\partial x^i}\frac{\partial}{\partial y^k}} V \\
=& \frac{\partial y^k}{\partial x^i}\frac{\partial b^j}{\partial y^k}\frac{\partial}{\partial y^j} \\
=& \frac{\partial y^k}{\partial x^i}\frac{\partial}{\partial y^k}(\frac{\partial y^j}{\partial x^m}a^m)\frac{\partial x^l}{\partial y^j}\frac{\partial}{\partial x^l} \\
=& \frac{\partial y^k}{\partial x^i}\frac{\partial x^l}{\partial y^j}\left(a^m\frac{\partial}{\partial y^k}\left(\frac{\partial y^j}{\partial x^m}\right) + \frac{\partial a^m}{\partial y^k}\frac{\partial y^j}{\partial x^m}\right)\frac{\partial}{\partial x^l} \\
=& \left(a^m\frac{\partial y^k}{\partial x^i}\frac{\partial x^l}{\partial y^j}\frac{\partial}{\partial y^k}\left(\frac{\partial y^j}{\partial x^m}\right) + \frac{\partial a^m}{\partial y^k}\frac{\partial y^k}{\partial x^i}\frac{\partial x^l}{\partial y^j}\frac{\partial y^j}{\partial x^m}\right)\frac{\partial}{\partial x^l} \\
=& \left(a^m\frac{\partial x^l}{\partial y^j}\frac{\partial^2 y^j}{\partial x^i \partial x^m} + \frac{\partial a^l}{\partial x^i}\right)\frac{\partial}{\partial x^l} \\
\neq& \frac{\partial a^l}{\partial x^i}\frac{\partial}{\partial x^l}.
\end{aligned}$$

2. Correction terms

So we want to add a correction term in the above so that the differentiation is the same for all coordinates: *general covariance*. Namely, define

$$D_{\frac{\partial}{\partial x^i}} V = \left(\frac{\partial a^k}{\partial x^i} + \Gamma^k_{ij}a^j\right)\frac{\partial}{\partial x^k}.$$

Similarly,

$$D_{\frac{\partial}{\partial y^i}} V = \left(\frac{\partial b^k}{\partial y^i} + \widetilde{\Gamma}^k_{ij}b^j\right)\frac{\partial}{\partial y^k}$$

So that

$$D_{\frac{\partial y^k}{\partial x^i}\frac{\partial}{\partial y^k}} V = D_{\frac{\partial}{\partial x^i}} V.$$

As in the previous computations:

$$
\begin{aligned}
\left(\frac{\partial a^l}{\partial x^i}+\Gamma_{ij}^l a^j\right)\frac{\partial}{\partial x^k} =& D_{\frac{\partial}{\partial x^i}}V \\
=& D_{\frac{\partial y^k}{\partial x^i}\frac{\partial}{\partial y^k}}V \\
=& \frac{\partial y^k}{\partial x^i}\left(\frac{\partial b^j}{\partial y^k}+\widetilde{\Gamma}_{kl}^j b^l\right)\frac{\partial}{\partial y^j} \\
=& \left(a^m\frac{\partial x^l}{\partial y^j}\frac{\partial^2 y^j}{\partial x^i\partial x^m}+\frac{\partial a^l}{\partial x^i}\right)\frac{\partial}{\partial x^l}+b^l\widetilde{\Gamma}_{kl}^j\frac{\partial y^k}{\partial x^i}\frac{\partial}{\partial y^j} \\
=& \left(a^m\frac{\partial x^l}{\partial y^j}\frac{\partial^2 y^j}{\partial x^i\partial x^m}+\frac{\partial a^l}{\partial x^i}\right)\frac{\partial}{\partial x^l}+b^l\widetilde{\Gamma}_{kl}^j\frac{\partial y^k}{\partial x^i}\frac{\partial x^m}{\partial y^j}\frac{\partial}{\partial x^m} \\
=& \left(a^m\frac{\partial x^l}{\partial y^j}\frac{\partial^2 y^j}{\partial x^i\partial x^m}+\frac{\partial a^l}{\partial x^i}+\frac{\partial y^m}{\partial x^q}a^q\widetilde{\Gamma}_{km}^j\frac{\partial y^k}{\partial x^i}\frac{\partial x^l}{\partial y^j}\right)\frac{\partial}{\partial x^l} \\
=& \left(\frac{\partial a^l}{\partial x^i}+\Gamma_{ij}^l a^j\right)\frac{\partial}{\partial x^l}.
\end{aligned}
$$

So

$$\Gamma_{ij}^l a^j = a^m\frac{\partial x^l}{\partial y^j}\frac{\partial^2 y^j}{\partial x^i\partial x^m}+a^q\frac{\partial y^m}{\partial x^q}\frac{\partial y^k}{\partial x^i}\frac{\partial x^l}{\partial y^j}\widetilde{\Gamma}_{km}^j.$$

This is true for all V. So we may let $a^p=1, a^i=0$ if $i\neq p$. Then

$$\Gamma_{ip}^l=\frac{\partial x^l}{\partial y^j}\frac{\partial^2 y^j}{\partial x^i\partial x^p}+\frac{\partial y^m}{\partial x^p}\frac{\partial y^k}{\partial x^i}\frac{\partial x^l}{\partial y^j}\widetilde{\Gamma}_{km}^j \tag{2.1}$$

That is if one can find such Γ_{ij}^k which transform in this way, then D is well-defined. Note that we usually require $\Gamma_{ij}^k=\Gamma_{ji}^k$. Then we also have $\widetilde{\Gamma}_{km}^j=\widetilde{\Gamma}_{mk}^j$.

Remark: There are infinitely many ways to do this.

Suppose we have such kind of corrections. Let $\alpha(t)=(x^1(t),\ldots,x^n(t))$. Then $\alpha'(t)=\frac{dx^i}{dt}\frac{\partial}{\partial x^i}$. So

$$
\begin{aligned}
D_{\alpha'(t)}\alpha'(t) =& D_{\alpha'}\frac{dx^k}{dt}\frac{\partial}{\partial x^k} \\
=& \frac{d^2x^k}{dt^2}\frac{\partial}{\partial x^k}+\frac{\partial x^k}{dt}\Gamma_{jk}^l\frac{\partial x^l}{dt}\frac{\partial}{\partial x^l} \\
=& \left(\frac{d^2x^k}{dt^2}+\Gamma_{jl}^k\frac{dx^j}{dt}\frac{dx^l}{dt}\right)\frac{\partial}{\partial x^l}.
\end{aligned}
\tag{2.2}
$$

3. Geodesics

Assume near a point, the surface is close to Euclidean in some sense: that is we can find coordinates ξ^i, so that $g_{ij}=\delta_{ij}$ and the geodesic equation is of the form:

$$\frac{d^2\xi^i}{ds^2}=0, \tag{3.1}$$

where s is the arc length. In any other coordinates, x^i, we have by chain rule

$$
\begin{aligned}
0 =& \frac{d}{ds}\left(\frac{\partial \xi^i}{\partial x^j}\frac{dx^j}{ds}\right) \\
=& \frac{\partial \xi^i}{\partial x^j}\frac{d^2x^j}{ds^2} + \frac{\partial^2 \xi^i}{\partial x^j \partial x^k}\frac{dx^j}{ds}\frac{dx^k}{ds}
\end{aligned} \tag{3.2}
$$

Multiple both sides by $\frac{\partial x^l}{\partial \xi^i}$ and sum on i, we have

$$
0 = \frac{d^2x^l}{ds^2} + \frac{\partial x^l}{\partial \xi^i}\frac{\partial^2 \xi^i}{\partial x^j \partial x^k}\frac{dx^j}{ds}\frac{dx^k}{ds}. \tag{3.3}
$$

Compare with (2.2), let where

$$
\Gamma^l_{jk} = \frac{\partial x^l}{\partial \xi^i}\frac{\partial^2 \xi^i}{\partial x^j \partial x^k}. \tag{3.4}
$$

If in y coordinates, the Christoffel symbols are denoted by $\widetilde{\Gamma}$, then

$$
\begin{aligned}
\widetilde{\Gamma}^l_{jk} =& \frac{\partial y^l}{\partial \xi^a}\frac{\partial^2 \xi^a}{\partial y^j \partial y^k} \\
=& \frac{\partial y^l}{\partial x^e}\frac{\partial x^e}{\partial \xi^a}\frac{\partial}{\partial x^p}\left(\frac{\partial \xi^a}{\partial y^k}\right)\frac{\partial x^p}{\partial y^j} \\
=& \frac{\partial y^l}{\partial x^e}\frac{\partial x^e}{\partial \xi^a}\frac{\partial}{\partial x^p}\left(\frac{\partial \xi^a}{\partial x^q}\frac{\partial x^q}{\partial y^k}\right)\frac{\partial x^p}{\partial y^j} \\
=& \frac{\partial y^l}{\partial x^e}\frac{\partial x^p}{\partial y^j}\frac{\partial x^q}{\partial y^k}\frac{\partial x^e}{\partial \xi^a}\frac{\partial^2 \xi^a}{\partial x^p \partial x^q} + \frac{\partial y^d}{\partial x^e}\frac{\partial x^e}{\partial \xi^a}\frac{\partial \xi^a}{\partial x^q}\frac{\partial}{\partial x^p}\left(\frac{\partial x^q}{\partial y^k}\right)\frac{\partial x^p}{\partial y^j} \\
=& \frac{\partial y^l}{\partial x^e}\frac{\partial x^p}{\partial y^j}\frac{\partial x^q}{\partial y^k}\Gamma^e_{pq} + \frac{\partial y^l}{\partial x^q}\frac{\partial^2 x^q}{\partial y^j \partial y^k}.
\end{aligned}
$$

Or:

$$
\begin{aligned}
\Gamma^e_{pq} =& -\frac{\partial x^e}{\partial y^l}\frac{\partial y^j}{\partial x^p}\frac{\partial y^k}{\partial x^q}\frac{\partial y^l}{\partial x^s}\frac{\partial^2 x^s}{\partial y^j \partial y^k} + \frac{\partial x^e}{\partial y^l}\frac{\partial y^j}{\partial x^p}\frac{\partial y^k}{\partial x^q}\widetilde{\Gamma}^l_{jk} \\
=& -\frac{\partial y^j}{\partial x^p}\frac{\partial y^k}{\partial x^q}\frac{\partial^2 x^e}{\partial y^j \partial y^k} + \frac{\partial x^e}{\partial y^l}\frac{\partial y^j}{\partial x^p}\frac{\partial y^k}{\partial x^q}\widetilde{\Gamma}^l_{jk} \\
=& -\frac{\partial y^j}{\partial x^p}\frac{\partial^2 x^e}{\partial x^q \partial y^j} + \frac{\partial x^e}{\partial y^l}\frac{\partial y^j}{\partial x^p}\frac{\partial y^k}{\partial x^q}\widetilde{\Gamma}^l_{jk} \\
=& -\frac{\partial}{\partial x^q}\left(\frac{\partial y^j}{\partial x^p}\frac{\partial x^e}{\partial y^j}\right) + \frac{\partial^2 y^j}{\partial x^p \partial x^q}\frac{\partial x^e}{\partial y^j} + \frac{\partial x^e}{\partial y^l}\frac{\partial y^j}{\partial x^p}\frac{\partial y^k}{\partial x^q}\widetilde{\Gamma}^l_{jk} \\
=& \frac{\partial^2 y^j}{\partial x^p \partial x^q}\frac{\partial x^e}{\partial y^j} + \frac{\partial x^e}{\partial y^l}\frac{\partial y^j}{\partial x^p}\frac{\partial y^k}{\partial x^q}\widetilde{\Gamma}^l_{jk}
\end{aligned}
$$

We see that this is consistent with (2.1).

To find Γ^k_{ij} which is independent of ξ. We proceed as follows.

In the ξ coordinates, $g(\frac{\partial}{\partial \xi^i}, \frac{\partial}{\partial \xi^j}) = \delta_{ij}$. Hence in x coordinates.

$$g_{ab} = g(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}) = \frac{\partial \xi^c}{\partial x^c}\frac{\partial \xi^d}{\partial x^b}\delta_{cd}$$

In fact,

$$g_{ab,s} = (\frac{\partial^2 \xi^c}{\partial x^a \partial x^s}\frac{\partial \xi^d}{\partial x^b} + \frac{\partial \xi^c}{\partial x^a}\frac{\partial^2 \xi^d}{\partial x^b \partial x^s})\delta_{cd}$$

So

$$\begin{aligned} g_{ad,b} + g_{db,a} - g_{ab,d} =& (\frac{\partial^2 \xi^p}{\partial x^a \partial x^b}\frac{\partial \xi^q}{\partial x^d} + \underline{\underline{\frac{\partial \xi^p}{\partial x^a}\frac{\partial^2 \xi^q}{\partial x^d \partial x^b}}} + \frac{\partial^2 \xi^p}{\partial x^b \partial x^a}\frac{\partial \xi^q}{\partial x^d} + \underline{\frac{\partial \xi^p}{\partial x^b}\frac{\partial^2 \xi^q}{\partial x^d \partial x^a}})\delta_{pq} \\ & - (\underline{\frac{\partial^2 \xi^p}{\partial x^a \partial x^d}\frac{\partial \xi^q}{\partial x^b}} + \underline{\underline{\frac{\partial \xi^p}{\partial x^a}\frac{\partial^2 \xi^q}{\partial x^b \partial x^d}}})\delta_{pq} \\ =& 2\frac{\partial^2 \xi^p}{\partial x^a \partial x^b}\frac{\partial \xi^q}{\partial x^d}\delta_{pq} \end{aligned}$$

Now

$$g^{cd} = \frac{\partial x^c}{\partial \xi^s}\frac{\partial x^d}{\partial \xi^t}\delta^{st}.$$

Hence

$$\frac{1}{2}g^{cd}\left(g_{ad,b} + g_{db,a} - g_{ab,d}\right) = \frac{\partial x^c}{\partial \xi^d}\frac{\partial^2 \xi^d}{\partial x^a \partial x^b} = \Gamma^c_{ab}. \tag{3.5}$$

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