Assignment 4, Due Tuesday Oct 9, 2018

(1) Consider the Enneper's surface:

$$\mathbf{X}(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, v^2 - u^2\right).$$

Find the coefficients of the first fundamental form and the second fundamental form. Show that the mean curvature of the Enneper's surface is 0 everywhere. Find the Gaussian curvature.

(2) Let $M = \{(x,y,z)|\ z = x^2 + ky^2\}$, with $k > 0$. That is M is the image of $\mathbf{X}(x,y) = (x, y, x^2 + ky^2)$. Let $p = (0,0,0)$, the origin. Show that $e_1 = (1,0,0)$ and $e_2 = (0,1,0)$ form a basis of $T_p(M)$. Let $\mathbf{N}$ be the unit normal $\mathbf{X}_x \times \mathbf{X}_y / |\mathbf{X}_x \times \mathbf{X}_y|$. Find the matrix of $\mathcal{S}_p : T_p(M) \to T_p(M)$ with respect to the basis $e_1,\ e_2$. Find the Gaussian curvature and the mean curvature of M at the point p.

(3) Consider the tractrix Let $\alpha : (0, \frac{\pi}{2}) \to xz$-plane given by

$$\alpha(t) = \left(\sin\, t, 0, \cos\, t + \log\, \tan\, \frac{t}{2}\right).$$

Show that the Gaussian curvature of the surface of revolution obtained by rotating α about the z-axis is -1. The surface is called the *pseudosphere*.

(4) Let $\mathbf{X} : U \to \mathbb{R}^3$ be a regular surface patch with unit normal vector field $\mathbf{N}$ and let $M = \mathbf{X}(U)$. Let f be a smooth function on M which is nowhere zero, i.e. if $\mathbf{X}(u,v) = (x(u,v), y(u,v), z(u,v))$ then $f = f(\mathbf{X}(u,v))$ is a smooth function in u, v. Let $p \in M$ and let $\mathbf{v}_1$ and $\mathbf{v}_2$ form an orthonormal basis for $T_p(M)$ such that $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{N}$. Prove that the Gaussian curvature of M at p is given by:

$$K = \frac{\langle d(f\mathbf{N})(\mathbf{v}_1) \times d(f\mathbf{N})(\mathbf{v}_2), \mathbf{N}\rangle}{f^2}.$$

Note $d(f\mathbf{N})(\mathbf{v})$ is defined as follow: let α be a smooth curve on M with $\alpha(0) = p$, $\alpha'(0) = \mathbf{v}$, then

$$d(f\mathbf{N})(\mathbf{v}) = \frac{d}{dt}(f(\alpha(t))\mathbf{N}(\alpha(t)))\big|_{t=0}.$$

(*You do not need to prove that the this is well-defined*).