Assignment 1, Due 13/9/2018

(1) (The tractrix) Let $\alpha : (0, \pi) \to \mathbb{R}^2$ be given by

$$\alpha(t) = \left(\sin\, t, \cos\, t + \log\, \tan \frac{t}{2} \right).$$

(a) Prove that α is regular except at $t = \frac{\pi}{2}$.

(b) Prove that the length of the segment of the tangent of α between the point of tangency and the y-axis is constantly 1.

(2) A regular curve $\alpha(s)$ parametrized by arc length is called a *cylindrical helix* if the is some constant vector $\mathbf{u}$ such that $\langle T, \mathbf{u} \rangle = \cos \theta_0$ is a constant. Prove that a regular curve α parametrized by arc length with $\kappa > 0$ is a cylindrical helix if and only if κ/τ is constant.

(3) Assume that $k(s) > 0$, $\tau(s) \neq 0$ and $k'(s) \neq 0$ for all s for a regular curve $\alpha(s)$ parametrized by arc length. Show that α lies on a sphere if and only if

$$\rho^2 + (\rho')^2 \lambda^2 = \text{constant}.$$

where $\rho = 1/k(s)$, $\lambda = 1/\tau$.

(*Hint*: Necessity: Differentiate $|\alpha|^2$ three times to obtain $\alpha = -\rho N - \rho' \lambda B$. Sufficiency: Show that $\beta = \alpha + \rho N - \rho' \Gamma B$ is constant.)